

# PURSuing THE DOUBLE AFFINE GRASSMANNIAN III: CONVOLUTION WITH AFFINE ZASTAVA

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ABSTRACT. This is the third paper of a series (started by [4], [5]) which describes a conjectural analog of the affine Grassmannian for affine Kac-Moody groups (also known as the double affine Grassmannian). The current paper is dedicated to describing a conjectural analog of the convolution diagram for the double affine Grassmannian and affine Zastava.

## 1. INTRODUCTION

**1.1. The usual affine Grassmannian.** Let  $G$  be a connected complex reductive group with a Cartan torus  $T$ , and let  $\mathcal{K} = \mathbb{C}((s))$ ,  $\mathcal{O} = \mathbb{C}[[s]]$ . By the *affine Grassmannian* of  $G$  we shall mean the quotient  $\mathrm{Gr}_G = G(\mathcal{K})/G(\mathcal{O})$ . It is known (cf. [1, 17]) that  $\mathrm{Gr}_G$  is the set of  $\mathbb{C}$ -points of an ind-scheme over  $\mathbb{C}$ , which we will denote by the same symbol. Note that  $\mathrm{Gr}_G$  is defined for any (not necessarily reductive) group  $G$ .

Let  $\Lambda = \Lambda_G$  denote the coweight lattice of  $G$  and let  $\Lambda^\vee$  denote the dual lattice (this is the weight lattice of  $G$ ). We let  $2\rho_G^\vee$  denote the sum of the positive roots of  $G$ .

The group-scheme  $G(\mathcal{O})$  acts on  $\mathrm{Gr}_G$  on the left and its orbits can be described as follows. One can identify the lattice  $\Lambda_G$  with the quotient  $T(\mathcal{K})/T(\mathcal{O})$ . Fix  $\lambda \in \Lambda_G$  and let  $s^\lambda$  denote any lift of  $\lambda$  to  $T(\mathcal{K})$ . Let  $\mathrm{Gr}_G^\lambda$  denote the  $G(\mathcal{O})$ -orbit of  $s^\lambda$  (this is clearly independent of the choice of  $s^\lambda$ ). The following result is well-known:

**Lemma 1.2.** (1)

$$\mathrm{Gr}_G = \bigcup_{\lambda \in \Lambda_G} \mathrm{Gr}_G^\lambda.$$

(2) We have  $\mathrm{Gr}_G^\lambda = \mathrm{Gr}_G^\mu$  if and only if  $\lambda$  and  $\mu$  belong to the same  $W$ -orbit on  $\Lambda_G$  (here  $W$  is the Weyl group of  $G$ ). In particular,

$$\mathrm{Gr}_G = \bigsqcup_{\lambda \in \Lambda_G^+} \mathrm{Gr}_G^\lambda.$$

(3) For every  $\lambda \in \Lambda^+$  the orbit  $\mathrm{Gr}_G^\lambda$  is finite-dimensional and its dimension is equal to  $\langle \lambda, 2\rho_G^\vee \rangle$ .

Let  $\overline{\mathrm{Gr}_G}^\lambda$  denote the closure of  $\mathrm{Gr}_G^\lambda$  in  $\mathrm{Gr}_G$ ; this is an irreducible projective algebraic variety; one has  $\mathrm{Gr}_G^\mu \subset \overline{\mathrm{Gr}_G}^\lambda$  if and only if  $\lambda - \mu$  is a sum of positive roots of the Langlands dual group  $G^\vee$ . We will denote by  $\mathrm{IC}^\lambda$  the intersection cohomology complex on  $\overline{\mathrm{Gr}_G}^\lambda$ . Let  $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$  denote the category of  $G(\mathcal{O})$ -equivariant perverse sheaves on  $\mathrm{Gr}_G$ . It is known that every object of this category is a direct sum of the  $\mathrm{IC}^\lambda$ 's.

**1.3. Transversal slices.** Consider the group  $G[s^{-1}] \subset G((s))$ ; let us denote by  $G[s^{-1}]_1$  the kernel of the natural (“evaluation at  $\infty$ ”) homomorphism  $G[s^{-1}] \rightarrow G$ . For any  $\lambda \in \Lambda$  let  $\text{Gr}_{G,\lambda} = G[s^{-1}] \cdot s^\lambda$ . Then it is easy to see that one has

$$\text{Gr}_G = \bigsqcup_{\lambda \in \Lambda^+} \text{Gr}_{G,\lambda}$$

Let also  $\mathcal{W}_{G,\lambda}$  denote the  $G[s^{-1}]_1$ -orbit of  $s^\lambda$ . For any  $\lambda, \mu \in \Lambda^+$ ,  $\lambda \geq \mu$  set

$$\text{Gr}_{G,\mu}^\lambda = \text{Gr}_G^\lambda \cap \text{Gr}_{G,\mu}, \quad \overline{\text{Gr}}_{G,\mu}^\lambda = \overline{\text{Gr}}_G^\lambda \cap \text{Gr}_{G,\mu}$$

and

$$\mathcal{W}_{G,\mu}^\lambda = \text{Gr}_G^\lambda \cap \mathcal{W}_{G,\mu}, \quad \overline{\mathcal{W}}_{G,\mu}^\lambda = \overline{\text{Gr}}_G^\lambda \cap \mathcal{W}_{G,\mu}.$$

Note that  $\overline{\mathcal{W}}_{G,\mu}^\lambda$  contains the point  $s^\mu$  in it. The variety  $\overline{\mathcal{W}}_{G,\mu}^\lambda$  can be thought of as a transversal slice to  $\text{Gr}_G^\mu$  inside  $\overline{\text{Gr}}_G^\lambda$  at the point  $s^\mu$  (cf. [4], Lemma 2.9).

**1.4. The convolution.** We can regard  $G(\mathcal{K})$  as a total space of a  $G(\mathcal{O})$ -torsor over  $\text{Gr}_G$ . In particular, by viewing another copy of  $\text{Gr}_G$  as a  $G(\mathcal{O})$ -scheme, we can form the associated fibration

$$\text{Gr}_G \star \text{Gr}_G := G(\mathcal{K}) \times_{G(\mathcal{O})} \text{Gr}_G = G(\mathcal{K}) \times_{G(\mathcal{O})} G(\mathcal{K})/G(\mathcal{O}).$$

One has the natural maps  $p, m : \text{Gr}_G \star \text{Gr}_G \rightarrow \text{Gr}_G$  defined as follows. Let  $g \in G(\mathcal{K}), x \in \text{Gr}_G$ . Then

$$p(g \times x) = g \bmod G(\mathcal{O}); \quad m(g \times x) = g \cdot x.$$

For any  $\lambda_1, \lambda_2 \in \Lambda_G^+$  let us set  $\text{Gr}_G^{\lambda_1} \star \text{Gr}_G^{\lambda_2}$  to be the corresponding subscheme of  $\text{Gr}_G \star \text{Gr}_G$ ; this is a fibration over  $\text{Gr}_G^{\lambda_1}$  with the typical fiber  $\text{Gr}_G^{\lambda_2}$ . Its closure is  $\overline{\text{Gr}}_G^{\lambda_1} \star \overline{\text{Gr}}_G^{\lambda_2}$ . In addition, we define

$$(\text{Gr}_G^{\lambda_1} \star \text{Gr}_G^{\lambda_2})^{\lambda_3} = m^{-1}(\text{Gr}_G^{\lambda_3}) \cap (\text{Gr}_G^{\lambda_1} \star \text{Gr}_G^{\lambda_2}).$$

It is known (cf. [16]) that

$$\dim((\text{Gr}_G^{\lambda_1} \star \text{Gr}_G^{\lambda_2})^{\lambda_3}) = \langle \lambda_1 + \lambda_2 + \lambda_3, \rho_G^\vee \rangle. \quad (1.1)$$

(It is easy to see that although  $\rho_G^\vee \in \frac{1}{2}\Lambda_G^\vee$ , the RHS of (1.1) is an integer whenever the above intersection is non-empty.)

Starting from any perverse sheaf  $\mathcal{T}$  on  $\text{Gr}_G$  and a  $G(\mathcal{O})$ -equivariant perverse sheaf  $\mathcal{S}$  on  $\text{Gr}_G$ , we can form their twisted external product  $\mathcal{T} \widetilde{\boxtimes} \mathcal{S}$  (see e.g. Section 4 of [17]), which will be a perverse sheaf on  $\text{Gr}_G \star \text{Gr}_G$ . For two objects  $\mathcal{S}_1, \mathcal{S}_2 \in \text{Perv}_{G(\mathcal{O})}(\text{Gr}_G)$  we define their convolution

$$\mathcal{S}_1 \star \mathcal{S}_2 = m_!(\mathcal{S}_1 \widetilde{\boxtimes} \mathcal{S}_2).$$

The following theorem, which is a categorical version of the Satake equivalence, is a starting point for this paper, cf. [16], [14] and [17]. The best reference so far is [1], Sect. 5.3.

**Theorem 1.5.** (1) Let  $\mathcal{S}_1, \mathcal{S}_2 \in \text{Perv}_{G(\mathcal{O})}(\text{Gr}_G)$ . Then  $\mathcal{S}_1 \star \mathcal{S}_2 \in \text{Perv}_{G(\mathcal{O})}(\text{Gr}_G)$ .  
(2) The convolution  $\star$  extends to a structure of a tensor category on  $\text{Perv}_{G(\mathcal{O})}(\text{Gr}_G)$ .

- (3) As a tensor category,  $\text{Perv}_{G(\mathfrak{o})}(\text{Gr}_G)$  is equivalent to the category  $\text{Rep}(G^\vee)$ . Under this equivalence, the object  $\text{IC}^\lambda$  goes over to the irreducible representation  $L(\lambda)$  of  $G^\vee$  with highest weight  $\lambda$ ).

**1.6.** The equivalence  $\text{Perv}_{G(\mathfrak{o})}(\text{Gr}_G) \xrightarrow{\sim} \text{Rep}(G^\vee)$  is given by a fiber functor [1], [17] of integration over semiinfinite orbits. Namely, let  $N_- \subset G$  be the unipotent radical of the negative Borel subgroup, and let  $\mathfrak{T}_\lambda \subset \text{Gr}_G$  be the orbit of  $N_-(\mathcal{K})$  through the point  $s^\lambda \in \text{Gr}_G$ . Then the weight  $\lambda$  component of the fiber functor is given by the cohomology with supports in  $\mathfrak{T}_\lambda$ . Let us recall an equivalent construction of this fiber functor.

From now on we assume that  $G$  is almost simple simply connected. We consider a smooth curve  $\mathbf{C}$  of genus 0 with two marked points  $0, \infty$ . Let  $\text{Bun}_G$  (resp.  $\text{Bun}_B$ ) stand for the moduli stack of  $G$ -bundles (resp.  $B$ -bundles) on  $\mathbf{C}$ . Here  $B$  is the positive Borel subgroup of  $G$ . The natural morphism  $\text{Bun}_B \rightarrow \text{Bun}_G$  is *not* proper, and Drinfeld has discovered a natural relative compactification  $\overline{\text{Bun}}_B$  of  $\text{Bun}_B$ . It is the moduli stack of the following data:

- (a) A  $G$ -bundle  $\mathcal{F}_G$  on  $\mathbf{C}$ ;
- (b) For each dominant weight  $\check{\lambda}$  of  $G$ , an invertible subsheaf  $\mathcal{L}^{\check{\lambda}} \subset \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}$ . Here  $V^{\check{\lambda}}$  stands for the irreducible  $G$ -module with highest weight  $\check{\lambda}$ , and  $\mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}$  stands for the associated vector bundle on  $\mathbf{C}$ .

The collection of invertible subsheaves  $\mathcal{L}^{\check{\lambda}} \subset \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}$  should satisfy the *Plücker relations*, that is, for any dominant weights  $\check{\lambda}$  and  $\check{\mu}$ , the tensor product  $\mathcal{L}^{\check{\lambda}} \otimes \mathcal{L}^{\check{\mu}} \subset \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}} \otimes \mathcal{V}_{\mathcal{F}_G}^{\check{\mu}}$  should coincide with  $\mathcal{L}^{\check{\lambda}+\check{\mu}} \subset \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}+\check{\mu}}$  under the natural direct summand embedding  $\mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}+\check{\mu}} \hookrightarrow \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}} \otimes \mathcal{V}_{\mathcal{F}_G}^{\check{\mu}}$ .

The connected components of  $\overline{\text{Bun}}_B$  are numbered by the coweights  $\lambda \in \Lambda$ : for  $(\mathcal{L}^{\check{\lambda}}) \in \overline{\text{Bun}}_B^\lambda$  we have  $\deg \mathcal{L}^{\check{\lambda}} = -\langle \lambda, \check{\lambda} \rangle$ .

We will denote by  $\Lambda^{\text{pos}} \subset \Lambda = \Lambda_G$  the cone of nonnegative linear combinations of positive coroots of  $G$ . For every  $\alpha \in \Lambda^{\text{pos}}$  we consider the closed embedding  $i_\alpha : \overline{\text{Bun}}_B \hookrightarrow \overline{\text{Bun}}_B$  given by sending  $(\mathcal{F}_G, \mathcal{L}^{\check{\lambda}} \subset \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}})$  to  $(\mathcal{F}_G, \mathcal{L}^{\check{\lambda}}(-\langle \alpha, \check{\lambda} \rangle \cdot 0) \subset \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}})$ .

Now let  $\overline{\mathcal{H}}_0^\lambda \rightarrow \text{Bun}_G \times \text{Bun}_G$  stand for the Hecke correspondence at the point  $0 \in \mathbf{C}$ : the pairs of  $G$ -bundles together with an isomorphism off  $0 \in \mathbf{C}$  whose pole at  $0 \in \mathbf{C}$  has order less than or equal to  $\lambda$ . The fibers of the projection  $p_1$  (resp.  $p_2$ ) of  $\overline{\mathcal{H}}_0^\lambda$  to the first (resp. second) copy of  $\text{Bun}_G$  are both isomorphic to  $\overline{\text{Gr}}_G^\lambda$ .

We define the Hecke correspondence  $(p, \phi) : \overline{\mathcal{G}}_0^\lambda = \overline{\mathcal{H}}_0^\lambda \times_{\text{Bun}_G} \overline{\text{Bun}}_B \rightarrow \overline{\text{Bun}}_B \times \overline{\text{Bun}}_B$ . It is the moduli stack of the following data:

- (a) a pair of  $G$ -bundles  $\mathcal{F}_G$  and  $\mathcal{F}'_G$  together with an isomorphism off  $0 \in \mathbf{C}$  lying in  $\overline{\mathcal{H}}_0^\lambda$ ;
- (b) For each dominant weight  $\check{\lambda}$  of  $G$ , an invertible subsheaf  $\mathcal{L}^{\check{\lambda}} \subset \mathcal{V}_{\mathcal{F}'_G}^{\check{\lambda}}$  satisfying the Plücker relations.

Forgetting the datum of  $\mathcal{F}_G$  defines the morphism  $p : \overline{\mathcal{G}}_0^\lambda \rightarrow \overline{\text{Bun}}_B$ . The morphism  $\phi : \overline{\mathcal{G}}_0^\lambda \rightarrow \overline{\text{Bun}}_B$  is defined as follows. The condition  $(\mathcal{F}_G, \mathcal{F}'_G) \in \overline{\mathcal{H}}_0^\lambda$  implies  $\mathcal{V}_{\mathcal{F}'_G}^{\check{\lambda}} \subset \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}(\langle -w_0\lambda, \check{\lambda} \rangle \cdot 0)$  for every dominant weight  $\check{\lambda}$ . Hence  $\mathcal{L}^{\check{\lambda}}(\langle w_0\lambda, \check{\lambda} \rangle \cdot 0) \subset \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}}$ , and we set  $\phi(\mathcal{F}_G, \mathcal{F}'_G, \mathcal{L}^{\check{\lambda}} \subset \mathcal{V}_{\mathcal{F}'_G}^{\check{\lambda}}) := (\mathcal{F}_G, \mathcal{L}^{\check{\lambda}}(\langle w_0\lambda, \check{\lambda} \rangle \cdot 0) \subset \mathcal{V}_{\mathcal{F}_G}^{\check{\lambda}})$ .

Finally, we are able to state a theorem (see [8] Theorem 3.1.4 and [12] Theorem 13.2) providing a version of the fiber functor from the category  $\text{Perv}_{G(\mathcal{O})}(\text{Gr}_G)$  to  $\text{Rep}(G^\vee)$ . For a finite-dimensional  $G^\vee$ -module  $V$ , and  $\mu \in \Lambda_G$ , we denote the  $\mu$ -weight subspace of  $V$  by  $V(\mu)$ .

**Theorem 1.7.**  $\phi_! \text{IC}(\overline{\mathcal{G}}_0^\lambda) \simeq \bigoplus_{\alpha \in \Lambda_G^{\text{pos}}} i_{\alpha!} \text{IC}(\overline{\text{Bun}}_B) \otimes V^\lambda(w_0(\lambda) + \alpha)$ .

**1.8.** The goal of the present paper is to formulate an analogue of Theorem 1.7 for the double affine Grassmannian. However, as we have seen in [4], [5], the affine versions of the objects like  $\overline{\text{Gr}}_G^\lambda$  or  $\overline{\text{Bun}}_B$  are out of reach at the moment being “too global”, and have to be replaced by certain transversal slices.

A transversal slice to the closed embedding  $i_\alpha : \overline{\text{Bun}}_B \hookrightarrow \overline{\text{Bun}}_B$  is a well known *Drinfeld Zastava* space  $Z^\alpha$  (see [12], [7]). It is defined as the moduli scheme of collections of invertible subsheaves  $\mathcal{L}^\lambda \subset \mathcal{O}_{\mathbf{C}} \otimes V^\lambda$  satisfying the Plücker relations, the degree conditions  $\deg \mathcal{L}^\lambda = -\langle \alpha, \check{\lambda} \rangle$ , and the conditions at  $\infty \in \mathbf{C}$ : each  $\mathcal{L}^\lambda \subset \mathcal{O}_{\mathbf{C}} \otimes V^\lambda$  is a *line subbundle* near  $\infty \in \mathbf{C}$ , and the fiber  $\mathcal{L}^\lambda|_\infty$  coincides with the highest line in  $V^\lambda$ .

By construction, we have a locally closed embedding  $z_\alpha : Z^\alpha \hookrightarrow \overline{\text{Bun}}_B$ , and we define the scheme  $\overline{\mathcal{G}} Z^{\lambda, \alpha}$  as the cartesian product of  $z_\alpha : Z^\alpha \hookrightarrow \overline{\text{Bun}}_B$  and  $\phi : \overline{\mathcal{G}}_0^\lambda \rightarrow \overline{\text{Bun}}_B$ .

For any  $\beta \leq \alpha \in \Lambda_G^{\text{pos}}$  we also have a closed embedding  $i_\beta^\alpha : Z^{\alpha-\beta} \hookrightarrow Z^\alpha$  which sends a collection  $(\mathcal{L}^\lambda \subset \mathcal{O}_{\mathbf{C}} \otimes V^\lambda)$  to a collection  $(\mathcal{L}^\lambda(-\langle \beta, \check{\lambda} \rangle \cdot 0) \subset \mathcal{O}_{\mathbf{C}} \otimes V^\lambda)$ .

Now Theorem 1.7 can be equivalently formulated as follows (see Theorem 13.2 of [12]):

**Theorem 1.9.**  $\phi_! \text{IC}(\overline{\mathcal{G}} Z^{\lambda, \alpha}) \simeq \bigoplus_{\beta \leq \alpha} i_{\beta!}^\alpha \text{IC}(Z^{\alpha-\beta}) \otimes V^\lambda(\lambda - \beta)$ .

The key observation underlying the proof of the theorem is that  $\phi^{-1}(i_\alpha^\alpha(0)) \cong \overline{\mathcal{T}}_{\lambda-\alpha} \cap \overline{\text{Gr}}_G^\lambda$ .

**1.10. The group  $G_{\text{aff}}$ .** From now on we assume that  $G$  is almost simple and simply connected. To a connected reductive group  $G$  as above one can associate the corresponding affine Kac-Moody group  $G_{\text{aff}}$  in the following way. One can consider the polynomial loop group  $G[t, t^{-1}]$  (this is an infinite-dimensional group ind-scheme)

It is well-known that  $G[t, t^{-1}]$  possesses a canonical central extension  $\tilde{G}$  of  $G[t, t^{-1}]$ :

$$1 \rightarrow \mathbb{G}_m \rightarrow \tilde{G} \rightarrow G[t, t^{-1}] \rightarrow 1.$$

Moreover,  $\tilde{G}$  has again a natural structure of a group ind-scheme.

The multiplicative group  $\mathbb{G}_m$  acts naturally on  $G[t, t^{-1}]$  and this action lifts to  $\tilde{G}$ . We denote the corresponding semi-direct product by  $G_{\text{aff}}$ ; we also let  $\mathfrak{g}_{\text{aff}}$  denote its Lie algebra.

The Lie algebra  $\mathfrak{g}_{\text{aff}}$  is an untwisted affine Kac-Moody Lie algebra. In particular, it can be described by the corresponding affine root system. We denote by  $\mathfrak{g}_{\text{aff}}^\vee$  the *Langlands dual affine Lie algebra* (which corresponds to the dual affine root system) and by  $G_{\text{aff}}^\vee$  the corresponding dual affine Kac-Moody group, normalized by the property that it contains  $G^\vee$  as a subgroup (cf. [4], Subsection 3.1 for more details).

We denote by  $\Lambda_{\text{aff}} = \mathbb{Z} \times \Lambda \times \mathbb{Z}$  the coweight lattice of  $G_{\text{aff}}$ ; this is the same as the weight lattice of  $G_{\text{aff}}^\vee$ . Here the first  $\mathbb{Z}$ -factor is responsible for the center of  $G_{\text{aff}}^\vee$  (or  $\widehat{G}^\vee$ ); it can also be thought of as coming from the loop rotation in  $G_{\text{aff}}$ . The second  $\mathbb{Z}$ -factor is responsible for the loop rotation in  $G_{\text{aff}}^\vee$  it may also be thought of as coming from the center of  $G_{\text{aff}}$ . We denote by  $\Lambda_{\text{aff}}^+$  the set of dominant weights of  $G_{\text{aff}}^\vee$  (which is the same as the

set of dominant coweights of  $G_{\text{aff}}$ ). We also denote by  $\Lambda_{\text{aff},k}$  the set of weights of  $G_{\text{aff}}^\vee$  of level  $k$ , i.e. all the weights of the form  $(k, \bar{\lambda}, n)$ . We put  $\Lambda_{\text{aff},k}^+ = \Lambda_{\text{aff}}^+ \cap \Lambda_{\text{aff},k}$ .

**Important notational convention:** From now on we shall denote elements of  $\Lambda$  by  $\bar{\lambda}, \bar{\mu} \dots$  (instead of just writing  $\lambda, \mu \dots$  in order to distinguish them from the coweights of  $G_{\text{aff}}$  (= weights of  $G_{\text{aff}}^\vee$ ), which we shall just denote by  $\lambda, \mu \dots$ ).

Let  $\Lambda_k^+ \subset \Lambda$  denote the set of dominant coweights of  $G$  such that  $\langle \bar{\lambda}, \alpha \rangle \leq k$  when  $\alpha$  is the highest root of  $\mathfrak{g}$ . Then it is well-known that a weight  $(k, \bar{\lambda}, n)$  of  $G_{\text{aff}}^\vee$  lies in  $\Lambda_{\text{aff},k}^+$  if and only if  $\bar{\lambda} \in \Lambda_k^+$  (thus  $\Lambda_{\text{aff},k} = \Lambda_k^+ \times \mathbb{Z}$ ).

Let also  $W_{\text{aff}}$  denote affine Weyl group of  $G$  which is the semi-direct product of  $W$  and  $\Lambda$ . It acts on the lattice  $\Lambda_{\text{aff}}$  (resp.  $\hat{\Lambda}$ ) preserving each  $\Lambda_{\text{aff},k}$  (resp. each  $\hat{\Lambda}_k$ ). In order to describe this action explicitly it is convenient to set  $W_{\text{aff},k} = W \ltimes k\Lambda$  which naturally acts on  $\Lambda$ . Of course the groups  $W_{\text{aff},k}$  are canonically isomorphic to  $W_{\text{aff}}$  for all  $k$ . Then the restriction of the  $W_{\text{aff}}$ -action to  $\Lambda_{\text{aff},k} \simeq \Lambda \times \mathbb{Z}$  comes from the natural  $W_{\text{aff},k}$ -action on the first multiple.

It is well known that every  $W_{\text{aff}}$ -orbit on  $\Lambda_{\text{aff},k}$  contains unique element of  $\Lambda_{\text{aff},k}^+$ . This is equivalent to saying that  $\Lambda_k^+ \simeq \Lambda/W_{\text{aff},k}$ .

**1.11.** Our main dream is to create an analog of the affine Grassmannian  $\text{Gr}_G$  and the above results about it in the case when  $G$  is replaced by the (infinite-dimensional) group  $G_{\text{aff}}$ . The first attempt to do so was made in [4]: namely, in *loc. cit.* we have constructed analogs of the varieties  $\overline{W}_{G,\mu}^\lambda$  in the case when  $G$  is replaced by  $G_{\text{aff}}$ . In [5], we constructed analogs of the varieties  $m_n^{-1}(\overline{W}_{G,\mu}^\lambda) \cap (\text{Gr}_G^{\lambda_1} \star \dots \star \text{Gr}_G^{\lambda_n})$  and  $m_n^{-1}(\overline{W}_{G,\mu}^\lambda) \cap (\overline{\text{Gr}}_G^{\lambda_1} \star \dots \star \overline{\text{Gr}}_G^{\lambda_n})$  (here  $\lambda = \lambda_1 + \dots + \lambda_n$ ) when  $G$  is replaced by  $G_{\text{aff}}$ . We have also constructed analogs of the corresponding pieces in the *Beilinson-Drinfeld Grassmannian* for  $G_{\text{aff}}$ .

We will denote by  $\Lambda_{\text{aff}}^{\text{pos}}$  the cone of nonnegative linear combinations of positive roots of  $G_{\text{aff}}^\vee$ . For  $\alpha \in \Lambda_{\text{aff}}^{\text{pos}}$  the affine Drinfeld Zastava space  $Z^\alpha$  was constructed in [3]. It is a certain closure of the space of degree  $\alpha$  based maps from  $(\mathbb{C}, \infty)$  to the Kashiwara flag scheme of  $G_{\text{aff}}$ . We also have parabolic versions  $Z_{G_{\text{aff}},P}^\theta$  of  $Z^\alpha = Z_{G_{\text{aff}},I}^\alpha$  ( $I$  stands for the Iwahori subgroup of  $G_{\text{aff}}$ ) which are certain closures of the spaces of based maps from  $(\mathbb{C}, \infty)$  to the Kashiwara parabolic flag schemes. Among those, the Uhlenbeck space  $\mathcal{U}_G^a(\mathbb{A}^2) = Z_{G_{\text{aff}},G[t]}^a$  stands out: it corresponds to the maximal parabolic containing all the finite simple roots.

Unfortunately, the definition of Zastava given in Section 1.8 produces in the affine case a scheme of *infinite type*  $\mathbf{Z}^\alpha$ . In the maximal parabolic case, the Uhlenbeck space  $\mathcal{U}_G^a(\mathbb{A}^2)$  is a partial resolution of  $\mathbf{Z}_{G_{\text{aff}},G[t]}^a$ . We have a natural forgetting morphism  $\mathbf{Z}^\alpha \rightarrow \mathbf{Z}_{G_{\text{aff}},G[t]}^a$  where  $a$  is the coefficient of the affine simple root in  $\alpha$ , and  $Z^\alpha$  is defined as the cartesian product of  $\mathbf{Z}^\alpha$  and  $\mathcal{U}_G^a(\mathbb{A}^2)$  over  $\mathbf{Z}_{G_{\text{aff}},G[t]}^a$ . It is an affine scheme of finite type.

The disadvantage of the above definition is that  $Z^\alpha$  does not solve any moduli problem, and hence is very cumbersome to work with. However, in the case  $G = \text{SL}(N)$ , the Zastava space  $Z^\alpha$  possesses a semismall resolution of singularities  $\mathcal{P}^\alpha$ , an *affine Laumon space* [11], which is a moduli space of parabolic sheaves on  $\mathbb{C} \times \mathbb{P}^1$ . Moreover, according to [13],  $\mathcal{P}^\alpha$  admits a realization as a quiver variety, i.e. as a certain GIT quotient. The corresponding categorical quotient  $\mathfrak{Z}^\alpha$  is an affine reduced irreducible normal scheme equipped with a morphism to  $Z^\alpha$  which is an isomorphism at the level of  $\mathbb{C}$ -points (and conjecturally just

an isomorphism). Since in this paper we are interested in topological questions only (such as stalks of IC sheaves), we can replace  $Z^\alpha$  by  $\mathfrak{Z}^\alpha$  for all practical purposes.

**1.12.** The main result of the present paper is a construction of an affine version of the scheme  $\overline{\mathcal{G}}Z^{\lambda,\alpha}$  equipped with a morphism  $\phi$  to the affine Zastava  $Z^\alpha$ . It is constructed as a quiver variety in the case  $G = \mathrm{SL}(N)$ , and then for general  $G$  via the adjoint embedding  $G \hookrightarrow \mathrm{SL}(\mathfrak{g})$ . We conjecture that Theorem 1.9 holds true in the affine setting as well.

Although we cannot describe  $\overline{\mathcal{G}}Z^{\lambda,\alpha}$  as a solution of a moduli problem, its open subscheme  $\mathcal{G}Z^{\lambda,\alpha}$  does admit such a description. Let us first assume  $\lambda$  has level 1. We consider the projective plane  $\mathbb{P}^2$  with homogeneous coordinates  $[z_0 : z_1 : z_2]$  such that the line  $\ell_\infty$  “at infinity” is given by the equation  $z_0 = 0$ , while  $\mathbf{C} = \ell_0 \subset \mathbb{P}^2$  is given by the equation  $z_2 = 0$ . We consider the blowup  $\widehat{\mathbb{P}}^2$  at the origin ( $z_1 = z_2 = 0$ ), and keep the names  $\ell_\infty$  and  $\ell_0$  for the proper transforms of  $\ell_\infty$  and  $\ell_0$ . Then  $\mathcal{G}Z^{\lambda,\alpha}$  is the moduli space of  $G$ -bundles on  $\widehat{\mathbb{P}}^2$  equipped with a reduction to  $B$  along  $\ell_0$  framed at  $\ell_\infty$ . Note that even with this modular definition, the construction of projection  $\phi : \mathcal{G}Z^{\lambda,\alpha} \rightarrow Z^\alpha$  is rather nontrivial, cf. Section 3.8.

Let us now assume  $\lambda$  has level  $k$ . We consider the blowup  $\widehat{\mathbb{P}}_k^2$  of  $\mathbb{P}^2$  at the origin, but not at the maximal ideal of the origin this time; rather at the ideal generated locally by  $(z_1^k, z_2)$ . This blowup has an isolated singularity of Kleinian type  $A_{k-1}$  lying off the proper transforms of  $\ell_\infty$  and  $\ell_0$ . We consider the stacky resolution  $\widehat{\mathcal{S}}^k$  of  $\widehat{\mathbb{P}}_k^2$ . Then again  $\mathcal{G}Z^{\lambda,\alpha}$  is the moduli space of  $G$ -bundles on  $\widehat{\mathcal{S}}^k$  equipped with a reduction to  $B$  along  $\ell_0$  framed at  $\ell_\infty$ . The projection  $\phi : \mathcal{G}Z^{\lambda,\alpha} \rightarrow Z^\alpha$  is constructed in Section 4.7. Similarly to the finite dimensional case of Section 1.8, we have  $\phi^{-1}(i_\alpha^\alpha(0)) \cong \mathfrak{T}_{\lambda-\alpha} \cap \mathrm{Gr}_{G_{\mathrm{aff}}}^\lambda$ ; for the definition of the RHS and the proof of the isomorphism, see [10], Section 8, especially Proposition 8.7.

In the special case when  $\lambda - \alpha = \mu := (k, 0, 0)$ , we have an intermediate open subspace  $\mathcal{G}Z^{\lambda,\alpha} \subset \overline{\mathcal{W}}_{G_{\mathrm{aff}},\mu}^\lambda \subset \overline{\mathcal{G}}Z^{\lambda,\alpha}$ . In Section 3.2 of [6] we have defined the *repellents*  $\mathfrak{T}_\mu^e \subset \overline{\mathcal{W}}_{G_{\mathrm{aff}},\mu}^\lambda$ ; they were also considered in [20] under the name of *MV cycles*. We conjecture that the central fiber  $\phi^{-1}(i_\alpha^\alpha(0)) \cap \overline{\mathcal{W}}_{G_{\mathrm{aff}},\mu}^\lambda$  coincides with  $\mathfrak{T}_\mu^e$ , and we prove the inclusion  $\mathfrak{T}_\mu^e \subset \phi^{-1}(i_\alpha^\alpha(0)) \cap \overline{\mathcal{W}}_{G_{\mathrm{aff}},\mu}^\lambda$  in Proposition 5.4.

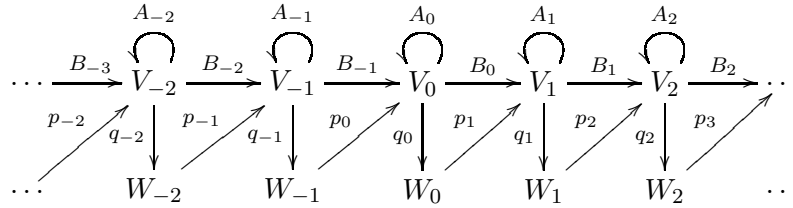
**1.13. Structure of the paper.** In Section 2 we recall the description of the affine Zastava  $Z_{\mathrm{SL}(N)_{\mathrm{aff}}}^\alpha$  in terms of representations of the *chainsaw quiver* of [13]. Contrary to the “global” approach of *loc. cit.*, we follow the classical ADHM approach on a 2-dimensional toric Deligne-Mumford stack  $\mathcal{S}_N = \mathbb{P}^1/\underline{\mu}_N \times \mathbb{P}^1$ . Here  $\underline{\mu}_N$  is the group of  $N$ -th roots of unity, acting on  $\mathbb{P}^1$  with fixed points  $0, \infty$ , and the quotient is categorical near  $\infty$ , and stacky near  $0$ . In Section 2.8–Section 2.11 we describe the irreducible components of the fixed point set  $(Z_{\mathrm{SL}(N)_{\mathrm{aff}}}^\alpha)^{\Gamma_k}$  of the action of a cyclic group  $\Gamma_k$ . In the central Section 3 we describe the parabolic torsion free sheaves on the blowup  $\widehat{\mathbb{P}}^2$  in terms of the *dented chainsaw quiver*  $\widehat{Q}$  (Section 3.1). The description is modeled on the one in [21] for torsion free sheaves on the blowup. The key Theorem 3.4 identifying the moduli space of parabolic torsion free sheaves on the blowup with a moduli space of  $\widehat{Q}$ -modules is due to A. Kuznetsov. We introduce the Zastava space for the blowup as the moduli space of  $\widehat{Q}$ -modules with certain stability conditions (Section 3.2). In Section 4 we introduce the Zastava space for the Kleinian blowup  $\widehat{\mathcal{S}}^k$  (Section 1.12) via a trick identifying it with a  $\Gamma_k$ -fixed points component in

the Zastava space on the blowup  $\widehat{\mathbb{P}}^2$ . This allows us to describe it as a moduli space of representations of the *rift quiver* (Section 4.3, Theorem 4.5). Finally, in Section 5, for an arbitrary almost simple simply connected group  $G$ , we define the Zastava space for the Kleinian blowup in terms of the one for  $\mathrm{SL}(\mathfrak{g})$ .

**1.14. Acknowledgments.** It is clear from the above that the paper owes its existence to A. Kuznetsov's generous explanations. We are grateful to him for the permission to reproduce his proof of the key Theorem 3.4. We are happy to thank the IAS at the Hebrew University of Jerusalem for the excellent working conditions. M. F. was partially supported by the RFBR grant 09-01-00242 and the Science Foundation of the SU-HSE awards No.T3-62.0 and 10-09-0015.

## 2. ZASTAVA AS A QUIVER VARIETY

**2.1. Chainsaw.** We recall some material from Section 2 of [13]. We consider the representations of the following *chainsaw quiver*  $Q$



with relations  $A_{l+1}B_l - B_lA_l + p_{l+1}q_l = 0 \forall l$ . Here the lower indices run through  $\mathbb{Z}/N\mathbb{Z}$ , and  $\dim V_l = d_l$ ,  $\dim W_l = 1$ . We denote by  $\underline{d}$  the collection of positive integers  $(d_l)_{l \in \mathbb{Z}/N\mathbb{Z}}$ . We denote by  $M_{\underline{d}}$  the scheme of representations of  $Q$ : a closed subscheme of

$$\bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} \mathrm{End}(V_l) \oplus \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}(V_l, V_{l+1}) \oplus \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}(W_{l-1}, V_l) \oplus \bigoplus_{l \in \mathbb{Z}/N\mathbb{Z}} \mathrm{Hom}(V_l, W_l)$$

given by equations  $A_{l+1}B_l - B_lA_l + p_{l+1}q_l = 0 \forall l$ . We denote by  $G_{\underline{d}}$  the group  $\prod_{l \in \mathbb{Z}/N\mathbb{Z}} \mathrm{GL}(V_l)$ ; it acts naturally on  $M_{\underline{d}}$ . We denote by  $\mathfrak{Z}_{\underline{d}}$  the categorical quotient  $M_{\underline{d}}/G_{\underline{d}}$ . According to Theorem 2.7 of [13],  $\mathfrak{Z}_{\underline{d}}$  is a reduced irreducible normal scheme which is the normalization of the affine Drinfeld Zastava space  $Z_{\mathrm{SL}(N)}^{\underline{d}}$  introduced in [3].

Furthermore, we consider an open subscheme  $M_{\underline{d}}^s \subset M_{\underline{d}}$  formed by all the *stable* representations of  $Q$ , i.e. those  $(A_l, B_l, p_l, q_l)_{l \in \mathbb{Z}/N\mathbb{Z}} \in M_{\underline{d}}$  such that there is no proper  $\mathbb{Z}/N\mathbb{Z}$ -graded subspace  $V'_\bullet \subset V_\bullet$  stable under  $A_\bullet, B_\bullet$  and containing  $p(W_\bullet)$ . Then the action of  $G_{\underline{d}}$  on  $M_{\underline{d}}^s$  is free, and the GIT quotient  $\mathfrak{M}_{\underline{d}} = M_{\underline{d}}^s/G_{\underline{d}}$  is a semismall resolution of  $\mathfrak{Z}_{\underline{d}}$ . Moreover, according to Section 2.3 of [13],  $\mathfrak{M}_{\underline{d}}$  is isomorphic to the moduli space  $\mathcal{P}_{\underline{d}}$  of torsion free parabolic sheaves of degree  $\underline{d}$  on a surface  $\mathbf{S}$ . Here  $\mathbf{S}$  is the product of two projective lines  $\mathbf{C}$  and  $\mathbf{X}$  with marked points  $0_{\mathbf{X}}, \infty_{\mathbf{X}} \in \mathbf{X}$  and  $0_{\mathbf{C}}, \infty_{\mathbf{C}} \in \mathbf{C}$ . The sheaves in question are equipped with a parabolic structure along a line  $\mathbf{D}_0 := \mathbf{C} \times 0_{\mathbf{X}}$ , and with a trivialization at “infinity”  $\mathbf{D}_\infty := \mathbf{C} \times \infty_{\mathbf{X}} \cup \infty_{\mathbf{C}} \times \mathbf{X}$ . The isomorphism  $\mathfrak{M}_{\underline{d}} \simeq \mathcal{P}_{\underline{d}}$  is deduced in *loc. cit.* from the “parabolic vs. orbifold” correspondence of [2] by global considerations. We will rephrase the argument in more local terms in Section 2.4 and Section 2.5 after some preparation in Section 2.2 and Section 2.3.

**2.2. ADHM.** To warm up we recall the classical ADHM construction (see e.g. Section 2 of [18]) following the approach of Section 5 of [9]. To this end we introduce the homogeneous coordinates  $(z : t)$  (resp.  $(y : x)$ ) on  $\mathbf{C}$  (resp.  $\mathbf{X}$ ) such that  $0_{\mathbf{C}}$  (resp.  $0_{\mathbf{X}}$ ) is given by  $z = 0$  (resp.  $y = 0$ ), and  $\infty_{\mathbf{C}}$  (resp.  $\infty_{\mathbf{X}}$ ) is given by  $t = 0$  (resp.  $x = 0$ ). The ADHM construction goes as follows. We consider the vector spaces  $V = \mathbb{C}^d$ ,  $W = \mathbb{C}^N$ , and the subscheme  $M_{N,d} \subset \text{End}(V) \oplus \text{End}(V) \oplus \text{Hom}(W, V) \oplus \text{Hom}(V, W)$  cut out by the equation  $AB - BA + pq = 0$  ( $A, B \in \text{End}(V)$ ,  $p \in \text{Hom}(W, V)$ ,  $q \in \text{Hom}(V, W)$ ). We consider an open subscheme  $M_{N,d}^s \subset M_{N,d}$  formed by all the *stable* quadruples  $(A, B, p, q)$ , i.e. such that  $V$  has no proper subspaces stable under  $A, B$  and containing  $p(W)$ . The group  $\text{GL}(V)$  acts naturally on  $M_{N,d}$ ; its action on  $M_{N,d}^s$  is free, and the GIT quotient  $M_{N,d}^s / \text{GL}(V)$  is denoted by  $\mathfrak{M}_{N,d}$ . It is isomorphic to the moduli space  $\mathcal{M}_{N,d}$  of torsion free sheaves of rank  $N$  and degree  $d$  on  $\mathbf{S}$  trivialized at  $\mathbf{D}_{\infty}$ . Namely,  $(A, B, p, q) \in \mathfrak{M}_{N,d}$  goes to the middle cohomology of the following monad of vector bundles on  $\mathbf{S}$ :

$$0 \rightarrow V \otimes \mathcal{O}_{\mathbf{S}}(-1, -1) \xrightarrow{C} V \otimes \mathcal{O}_{\mathbf{S}}(-1, 0) \xrightarrow{D} V \otimes \mathcal{O}_{\mathbf{S}} \rightarrow 0,$$

$$\begin{array}{c} \oplus \\ \oplus \\ W \otimes \mathcal{O}_{\mathbf{S}} \end{array}$$

$C = (tA - z, xB - y, txq)$ ,  $D = (-xB + y, tA - z, p)$ , where we write  $\mathcal{O}_{\mathbf{S}}(-1, -1)$  for  $\mathcal{O}_{\mathbf{C}}(-1) \boxtimes \mathcal{O}_{\mathbf{X}}(-1)$ , and  $\mathcal{O}_{\mathbf{S}}(0, -1)$  for  $\mathcal{O}_{\mathbf{C}} \boxtimes \mathcal{O}_{\mathbf{X}}(-1)$ , etc., and we view  $x, y$  (resp.  $z, t$ ) as a basis of  $\Gamma(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(1))$  (resp.  $\Gamma(\mathbf{C}, \mathcal{O}_{\mathbf{C}}(1))$ ).

**2.3. Stack  $\mathcal{S}_N$ .** We define a one-dimensional Deligne-Mumford stack  $\mathcal{X}_N$  as follows. Let  $\mathbf{X}_N \xrightarrow{\theta} \mathbf{X}$  denote the  $N$ -fold cyclic covering ramified over  $0_{\mathbf{X}}$  and  $\infty_{\mathbf{X}}$ . It is equipped with the action of the Galois group  $\Gamma_N \simeq \mathbb{Z}/N\mathbb{Z}$ . The action of  $\Gamma_N$  on  $\theta^{-1}(\mathbf{X} - 0_{\mathbf{X}} - \infty_{\mathbf{X}})$  is free, and the quotient is  $\mathbf{X} - 0_{\mathbf{X}} - \infty_{\mathbf{X}}$ . We glue the categorical quotient  $\theta^{-1}(\mathbf{X} - 0_{\mathbf{X}}) / \Gamma_N = \mathbf{X} - 0_{\mathbf{X}}$  with the stack quotient  $\theta^{-1}(\mathbf{X} - \infty_{\mathbf{X}}) / \Gamma_N$  over the common open substack  $\mathbf{X} - 0_{\mathbf{X}} - \infty_{\mathbf{X}}$  to obtain the desired stack  $\mathcal{X}_N$ . Note that  $\mathcal{X}_N$  is equipped with a projection  $\vartheta$  to  $\mathbf{X}$  which is an isomorphism off  $0_{\mathbf{X}}$ . The unique point of  $\mathcal{X}_N$  lying over  $0_{\mathbf{X}}$  will be denoted by  $0_{\mathcal{X}}$ ; its group of automorphisms is  $\Gamma_N$ . The unique point of  $\mathcal{X}_N$  lying over  $\infty_{\mathbf{X}}$  will be denoted by  $\infty_{\mathcal{X}}$ . Since  $N$  is fixed throughout the Section, we will omit the lower index  $N$  to simplify the notations.

We denote  $\mathcal{O}_{\mathcal{X}}(\pm \infty_{\mathcal{X}})$  by  $\mathcal{O}_{\mathcal{X}}(\pm N)$ . For  $0 \leq l \leq N$  we denote  $\mathcal{O}_{\mathcal{X}}(-l \cdot 0_{\mathcal{X}})$  by  $\mathcal{R}_l$ . Note that  $\mathcal{R}_N \simeq \mathcal{O}_{\mathcal{X}}(-N)$ . We have the canonical embeddings

$$\mathcal{R}_0(-N) \simeq \mathcal{R}_N \xrightarrow{\xi_N} \mathcal{R}_{N-1} \xrightarrow{\xi_{N-1}} \dots \xrightarrow{\xi_3} \mathcal{R}_2 \xrightarrow{\xi_2} \mathcal{R}_1 \xrightarrow{\xi_1} \mathcal{R}_0.$$

We define a 2-dimensional Deligne-Mumford stack  $\mathcal{S}_N$  as  $\mathbf{C} \times \mathcal{X}_N$ ; by an abuse of notation we will denote by  $\vartheta$  its projection  $\text{id} \times \vartheta$  onto  $\mathbf{S}$ . We denote  $\mathbf{C} \times 0_{\mathcal{X}}$  by  $\mathcal{D}_0$ , and we denote  $\infty_{\mathbf{C}} \times \mathcal{X} \cup \infty_{\mathcal{X}} \times \mathbf{C}$  by  $\mathcal{D}_{\infty}$ . By an abuse of notation, we denote by  $\mathcal{R}_l$  the line bundle  $\mathcal{O}_{\mathbf{C}} \boxtimes \mathcal{R}_l$ , and we denote by  $\xi_l$  the morphism  $\text{id} \boxtimes \xi_l$ . According to [2], there is a one-to-one correspondence between the (torsion free, framed at  $\mathcal{D}_{\infty}$ ) sheaves on  $\mathcal{S}$ , and the (torsion free, framed at  $\mathbf{D}_{\infty}$ ) sheaves on  $\mathbf{S}$  with parabolic structure along  $\mathbf{D}_0$ . Thus  $\mathcal{P}_{\underline{d}}$  is the moduli space of torsion free sheaves of degree  $\underline{d}$  on  $\mathcal{S}$  framed at  $\mathcal{D}_{\infty}$ .



**2.4. Monad for the stack  $\mathcal{S}_N$ .** Finally we are able to recall an ADHM-like construction of the isomorphism  $\mathfrak{M}_{\underline{d}} \xrightarrow{\sim} \mathcal{P}_{\underline{d}}$ . Note that  $\vartheta_*$  establishes an isomorphism  $\Gamma(\mathcal{X}, \mathcal{O}_{\mathcal{X}}(1)) \simeq \Gamma(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(1)) = \mathbb{C}\langle x, y \rangle$ . The desired isomorphism  $\mathfrak{M}_{\underline{d}} \xrightarrow{\sim} \mathcal{P}_{\underline{d}}$  sends a representative  $(A_{\bullet}, B_{\bullet}, p_{\bullet}, q_{\bullet})$  to the middle cohomology of the following monad of vector bundles on  $\mathcal{S}$ :

$$\bigoplus_{0 < l \leq N} V_l \otimes \mathcal{R}_l \otimes \mathcal{O}_{\mathcal{S}}(-1, 0) \xrightarrow{C} \bigoplus_{0 \leq l < N} V_{l+1} \otimes \mathcal{R}_l \otimes \mathcal{O}_{\mathcal{S}}(-1, 0) \xrightarrow{D} \bigoplus_{0 \leq l < N} V_{l+1} \otimes \mathcal{R}_l \quad (2.1)$$

$$\bigoplus_{0 \leq l < N} W_l \otimes \mathcal{R}_l$$

Here the “matrix coefficients” of  $C, D$  are as follows:  ${}_u C_{ll}^V = tA_l - z$ ;  ${}_u C_{l, l-1}^V = -\xi_l$ ;  ${}_{NN} C_{10}^V = xB_0$ , and  ${}_u C_{l+1, l}^V = B_l$  for  $0 < l < N$ ; furthermore,  ${}_{NN} C_{00}^W = txq_0$ , and  ${}_u C_{ll}^W = tq_l$  for  $0 < l < N$ . Furthermore,  ${}_{l+1, l} D_{l+1, l}^V = tA_l - z$ ;  ${}_u D_{l, l-1}^V = \xi_l$ ;  ${}_{NN} D_{10}^V = -xB_0$ , and  ${}_u D_{l+1, l}^V = -B_l$  for  $0 < l < N$ ; furthermore,  ${}_u D_{l+1, l}^W = p_{l+1}$ . We have used some evident shortcuts to simplify the notations, e.g.  ${}_{NN} C_{10}^V = xB_0 := B_0 \otimes x \otimes 1 \in \text{Hom}(V_0, V_1) \otimes \text{Hom}(\mathcal{R}_N, \mathcal{R}_0) \otimes \text{Hom}_{\mathbb{C}}(\mathcal{O}_{\mathbb{C}}(-1), \mathcal{O}_{\mathbb{C}}(-1))$ .

**2.5. Inverse construction.** Conversely, given a torsion free sheaf  $\mathcal{F}$  on  $\mathcal{S}_N$  framed at  $\mathcal{D}_{\infty}$ , and  $0 \leq l < N$ , we have (cf. Section 5 of [9])  $H^0(\mathcal{S}, \mathcal{R}_l^* \otimes \mathcal{F}(-1, 0)) = H^0(\mathcal{S}, \mathcal{R}_l^* \otimes \mathcal{F}(0, -N)) = H^0(\mathcal{S}, \mathcal{F}(-1, -N)) = 0$ ,  $H^2(\mathcal{S}, \mathcal{R}_l^* \otimes \mathcal{F}(-1, 0)) = H^2(\mathcal{S}, \mathcal{R}_l^* \otimes \mathcal{F}(0, -N)) = H^2(\mathcal{S}, \mathcal{F}(-1, -N)) = 0$ . Furthermore, for  $0 < l < N$ ,  $H^1(\mathcal{S}, \mathcal{R}_l^* \otimes \mathcal{F}(-1, 0)) \simeq H^1(\mathcal{S}, \mathcal{R}_l^* \otimes \mathcal{F}(0, -N)) \simeq V_l$ , and  $H^1(\mathcal{S}, \mathcal{F}(-1, 0)) \simeq H^1(\mathcal{S}, \mathcal{F}(0, -N)) \simeq H^1(\mathcal{S}, \mathcal{F}(-1, -N)) \simeq V_0$ . Furthermore, for  $0 \leq l < N - 1$ , we have a canonical exact sequence

$$0 \rightarrow H^0(\mathcal{S}, \mathcal{R}_l^* \otimes \mathcal{F}) \rightarrow W_l \xrightarrow{p_{l+1}} V_{l+1} \rightarrow H^1(\mathcal{S}, \mathcal{R}_{l+1}^* \otimes \mathcal{F}) \rightarrow 0,$$

and also

$$0 \rightarrow H^0(\mathcal{S}, \mathcal{R}_{N-1}^* \otimes \mathcal{F}) \rightarrow W_{N-1} \xrightarrow{p_0} V_0 \rightarrow H^1(\mathcal{S}, \mathcal{F}) \rightarrow 0,$$

The Beilinson spectral sequence for  $\mathcal{F}$  takes the form  $E_1^{i,j} =$

$$\bigoplus_{0 < l \leq N} \text{Ext}^j(\mathcal{R}_l(1, 0), \mathcal{F}) \otimes \mathcal{R}_l(-1, 0) \rightarrow \bigoplus_{0 \leq l < N} \text{Ext}^j(\mathcal{R}_{l+1}(1, 0), \mathcal{F}) \otimes \mathcal{R}_l(-1, 0) \rightarrow \bigoplus_{0 \leq l < N} \text{Ext}^j(\mathcal{R}_{l+1}, \mathcal{F}) \otimes \mathcal{R}_l,$$

( $i = -2, -1, 0$ ), that is

$$\bigoplus_{0 < l \leq N} V_l \otimes \mathcal{R}_l(-1, 0) \xrightarrow{\quad} \bigoplus_{0 < l \leq N} V_l \otimes \mathcal{R}_l \oplus \bigoplus_{0 \leq l < N} V_{l+1} \otimes \mathcal{R}_l(-1, 0) \xrightarrow{\quad} \bigoplus_{0 \leq l < N} H^1(\mathcal{R}_{l+1}^* \otimes \mathcal{F}) \otimes \mathcal{R}_l$$

$$\searrow d_2^{-2,1} \xrightarrow{\quad} \bigoplus_{0 \leq l < N} H^0(\mathcal{R}_l^* \otimes \mathcal{F}) \otimes \mathcal{R}_l$$

Finally, we can replace  $H^1(\mathcal{R}_{l+1}^* \otimes \mathcal{F}) = \text{Coker } p_{l+1}$  (resp.  $H^0(\mathcal{R}_l^* \otimes \mathcal{F}) = \text{Ker } p_{l+1}$ ) by  $V_{l+1}$  (resp.  $W_l$ ), and lift the differential  $d_2^{-2,1} : E_2^{-2,1} \rightarrow E_2^{0,0}$  to a morphism  $V_l \otimes \mathcal{R}_l(-1, 0) \rightarrow W_l \otimes \mathcal{R}_l$ . Replacing the spectral sequence with the total complex we obtain the ADHM description (2.1) of  $\mathcal{F}$ .

**2.6. Monad for the stack  $\mathcal{S}'_N$ .** We also consider the following version of the above construction. Let  $\mathcal{S}' = \mathcal{S}'_N$  be the stacky weighted projective plane  $\mathbb{P}^2(N, N, 1)$ . More precisely, we consider the affine 3-space  $\mathbb{A}^3$  with coordinates  $(z_0, z_1, z_2)$ , and with the action of  $\mathbb{C}^*$  given by  $c(z_0, z_1, z_2) = (c^N z_0, c^N z_1, c z_2)$ . We define  $\mathcal{S}' := (\mathbb{A}^3 \setminus 0)/\mathbb{C}^*$ . We define  $\ell \subset \mathcal{S}'$  as the hyperplane  $z_2 = 0$  (all the points of this line have automorphism group  $\mathbb{Z}/N\mathbb{Z}$ ), and we define  $\ell_\infty \subset \mathcal{S}'$  as the hyperplane  $z_0 = 0$ . Note that  $\ell_\infty \simeq \mathcal{X}$ . We denote  $\mathcal{O}_{\mathcal{S}'}(\ell)$  by  $\mathcal{O}(l)$  for short; note that  $\mathcal{O}_{\mathcal{S}'}(\ell_\infty) \simeq \mathcal{O}(N)$ . Let  $\mathcal{P}'_{\underline{d}}$  be the moduli space of torsion free sheaves of degree  $\underline{d}$  on  $\mathcal{S}'$  framed at  $\ell_\infty$ , i.e. such that  $\mathcal{F}|_{\ell_\infty} = \mathcal{F}_\infty := W_0 \otimes \mathcal{O}_{\mathcal{X}} \oplus W_1 \otimes \mathcal{O}_{\mathcal{X}}(-1) \oplus \dots \oplus W_{N-1} \otimes \mathcal{O}_{\mathcal{X}}(1-N)$ .

Since  $\mathcal{S} - \mathcal{D}_\infty \simeq \mathcal{S}' - \ell_\infty$ , and the framings at infinities match, we have an identification  $\mathcal{P}_{\underline{d}} \simeq \mathcal{P}'_{\underline{d}}$ . We describe the resulting isomorphism  $\mathfrak{M}_{\underline{d}} \xrightarrow{\sim} \mathcal{P}'_{\underline{d}}$ . It sends a representative  $(A_\bullet, B_\bullet, p_\bullet, q_\bullet)$  to the middle cohomology of the following monad of vector bundles on  $\mathcal{S}'$ :

$$\begin{array}{ccc} & \bigoplus_{0 < l \leq N} V_l(1-l) & \\ & \bigoplus & \\ \bigoplus_{0 < l \leq N} V_l(-l) & \xrightarrow{C} \bigoplus_{0 < l \leq N} V_l(N-l) & \xrightarrow{D} \bigoplus_{0 < l \leq N} V_l(N+1-l) \\ & \bigoplus & \\ & \bigoplus_{0 \leq l < N} W_l(-l) & \end{array} \quad (2.2)$$

Here the “matrix coefficients” of  $C, D$  are as follows:  $-z_2 : V_l(-l) \rightarrow V_l(1-l)$ ;  $z_0 B_0 : V_N(-N) \rightarrow V_1(0)$ ;  $B_l : V_l(-l) \rightarrow V_{l+1}(-l)$ ;  $z_1 - z_0 A_l : V_l(-l) \rightarrow V_l(N-l)$ ;  $z_0 q_0 : V_N(-N) \rightarrow W_0$ ;  $q_l : V_l(-l) \rightarrow W_l(-l)$ ; furthermore,  $z_1 - z_0 A_l : V_l(1-l) \rightarrow V_l(N+1-l)$ ;  $z_2 : V_l(N-l) \rightarrow V_l(N+1-l)$ ;  $-z_0 B_0 : V_N(0) \rightarrow V_1(N)$ ;  $-B_l : V_l(N-l) \rightarrow V_l(N-l) \rightarrow V_{l+1}(N-l)$ ;  $z_0 p_{l+1} : W_l(-l) \rightarrow V_{l+1}(N-l)$ .

**2.7. Rotation and the inverse construction.** Conversely, given a torsion free sheaf  $\mathcal{F}$  on  $\mathcal{S}'$  framed at  $\ell_\infty$ , and  $l = 1, \dots, N$ , we have  $H^0(\mathcal{S}', \mathcal{F}(l-N-1)) = H^2(\mathcal{S}', \mathcal{F}(l-N-1)) = 0$ , and  $V_l = H^1(\mathcal{S}', \mathcal{F}(l-N-1))$ . The endomorphisms  $A_l$  arise from the action of  $z_1 \in \Gamma(\mathcal{S}', \mathcal{O}(N))$ , and  $B_l$  arises from the action of  $z_2 \in \Gamma(\mathcal{S}', \mathcal{O}(1))$ . More precisely, for  $l \in \mathbb{Z}$ , we have the morphism  $z_2 : \mathcal{F}(l-N-1) \rightarrow \mathcal{F}(l-N)$  which induces  $B_l : V_l \rightarrow V_{l+1}$  for  $1 \leq l \leq N-1$ , and also  $z_2 : H^1(\mathcal{S}', \mathcal{F}(-N-1)) \rightarrow H^1(\mathcal{S}', \mathcal{F}(-N)) = V_1$ . However, the short exact  $0 \rightarrow \mathcal{F}(-N-1) \xrightarrow{z_0} \mathcal{F}(-1) \rightarrow \mathcal{F}_\infty(-1) \rightarrow 0$  gives rise to the long exact sequence of cohomology including  $z_0 : H^1(\mathcal{S}', \mathcal{F}(-N-1)) \xrightarrow{\sim} H^1(\mathcal{S}', \mathcal{F}(-1)) = V_N$ . So we define  $B_0 : V_N \rightarrow V_1$  as the composition  $z_2 z_0^{-1}$ .

Furthermore, we define  $A_0 : V_N \rightarrow V_N$  as the composition  $z_1 z_0^{-1}$ .

Furthermore, the short exact sequence  $0 \rightarrow \mathcal{F}(-N) \xrightarrow{z_0} \mathcal{F} \rightarrow \mathcal{F}_\infty \rightarrow 0$  gives rise to the long exact sequence of cohomology including  $W_0 = H^0(\mathcal{X}, \mathcal{F}_\infty) \rightarrow H^1(\mathcal{S}', \mathcal{F}(-N)) = V_1$ . We define  $p_1$  as this latter map  $W_0 \rightarrow V_1$ .

Furthermore, for  $0 \leq l < N$ , the short exact sequence

$$0 \rightarrow \mathcal{F}(l - 2N - 1) \xrightarrow{z_0} \mathcal{F}(l - N - 1) \rightarrow \mathcal{F}_\infty(l - N - 1) \rightarrow 0$$

gives rise to the long exact sequence of cohomology including  $H^1(\mathcal{S}', \mathcal{F}(l - N - 1)) \rightarrow H^1(\mathcal{X}, \mathcal{F}_\infty(l - N - 1)) = W_l \oplus W_{l+1} \oplus \dots \oplus W_{N-1}$ . For  $0 < l < N$ , we define  $q_l : V_l = H^1(\mathcal{S}', \mathcal{F}(l - N - 1)) \rightarrow W_l$  as the direct summand of the above morphism. For  $l = 0$ , we define  $q_0 : V_N = H^1(\mathcal{S}', \mathcal{F}(-1)) \rightarrow W_0$  as the composition of the direct summand of the above morphism with  $z_0^{-1} : V_N = H^1(\mathcal{S}', \mathcal{F}(-1)) \rightarrow H^1(\mathcal{S}', \mathcal{F}(-N - 1))$ .

It remains to define  $A_l$ ,  $l \neq 0$ , and  $p_l$ ,  $l \neq 1$ . To this end, we define the *rotation*  $\rho \underline{d}$  as follows:  $\rho d_l := d_{l+1}$ ,  $l \in \mathbb{Z}/N\mathbb{Z}$ . We have a natural *rotation* isomorphism  $R : \mathfrak{M}_{\underline{d}} \xrightarrow{\sim} \mathfrak{M}_{\rho \underline{d}}$ , taking the quiver data  $(V_\bullet, W_\bullet, A_\bullet, B_\bullet, p_\bullet, q_\bullet)$  to  $(V_{\bullet-1}, W_{\bullet-1}, A_{\bullet-1}, B_{\bullet-1}, p_{\bullet-1}, q_{\bullet-1})$ . We define the corresponding isomorphism  $R : \mathcal{P}'_{\underline{d}} \xrightarrow{\sim} \mathcal{P}'_{\rho \underline{d}}$  presently.

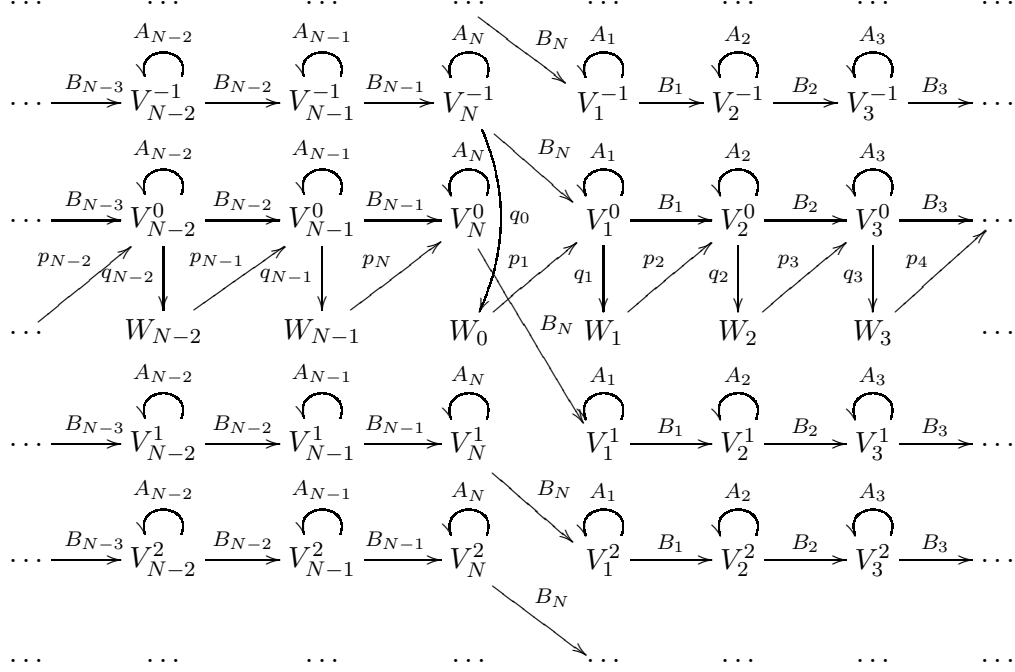
Given a framed torsion free sheaf  $\mathcal{F}$  on  $\mathcal{S}'$ , we define  $\mathcal{G} = R(\mathcal{F})$  as the kernel of the natural projection  $\mathcal{F}(1) \twoheadrightarrow \iota_* W_0(1)$ . Here  $\iota$  stands for the closed embedding  $\mathcal{X} \simeq \ell_\infty \hookrightarrow \mathcal{S}'$ . We have an exact sequence  $0 \rightarrow W_0(1 - N) \rightarrow \iota^* \mathcal{G} \xrightarrow{r} \iota^* \mathcal{F}(1) \rightarrow W_0(1) \rightarrow 0$ , and the morphism  $r$  factors as the composition  $\iota^* \mathcal{G} \rightarrow W_1 \oplus W_2(-1) \oplus \dots \oplus W_{N-1}(2 - N) \rightarrow \iota^* \mathcal{F}(1)$ . Since for any  $l = 1, \dots, N - 1$  we have  $\text{Ext}_{\mathcal{X}}^1(W_l(1 - l), W_0(1 - N)) = 0$ , we conclude that  $\mathcal{G}|_{\ell_\infty} \simeq W_1 \otimes \mathcal{O}_{\mathcal{X}} \oplus W_2(-1) \oplus \dots \oplus W_{N-1}(2 - N) \oplus W_0(1 - N)$ .

Furthermore, the long exact cohomology sequence arising from the short exact sequence  $0 \rightarrow \mathcal{G}(l - N - 1) \rightarrow \mathcal{F}(l - N) \rightarrow \iota_* W_0(l - N) \rightarrow 0$  implies  $H^1(\mathcal{S}', \mathcal{G}(l - N - 1)) = V_{l+1}$  for  $0 < l < N$ . Also, the long exact cohomology sequence arising from the short exact sequence  $0 \rightarrow \mathcal{F}(-N) \rightarrow \mathcal{G}(-1) \rightarrow \iota_*(W_1(-1) \oplus \dots \oplus W_{N-1}(1 - N)) \rightarrow 0$  implies  $H^1(\mathcal{S}', \mathcal{G}(-1)) = H^1(\mathcal{S}', \mathcal{F}(-N)) = V_1$ . Finally, it is clear that  $R^N = \text{Id} : \mathcal{P}'_{\underline{d}} \rightarrow \mathcal{P}'_{\underline{d}}$ .

Returning to the definition of  $A_l, p_l$ , we set  $A_l := R^{-l} A_0 R^l$ ,  $p_l := R^{1-l} p_1 R^{l-1}$ .

**2.8. The action of  $\Gamma_k$ .** Let  $\Gamma_k \simeq \mathbb{Z}/k\mathbb{Z}$  (resp.  $\Gamma_{kN} \simeq \mathbb{Z}/kN\mathbb{Z}$ ) be the group of  $k$ -th (resp.  $kN$ -th) roots of unity, with generator  $\zeta_k$  (resp.  $\zeta_{kN}$ ). We have a surjection  $\Gamma_{kN} \twoheadrightarrow \Gamma_k$ ,  $\zeta_{kN} \mapsto \zeta_k$ . The group  $\Gamma_k$  acts on  $\mathbf{S}$  as follows:  $\zeta_k(z : t, y : x) = (z : t, \zeta_k y : x)$ . Note that this action *does not* lift to the action of  $\Gamma_k$  on  $\mathcal{S}$ , it only lifts to the action of  $\Gamma_{kN}$  on  $\mathcal{S}$ . The corresponding action of  $\Gamma_{kN}$  on  $\mathcal{S}'$  is given by  $\zeta_{kN}(z_0, z_1, z_2) = (z_0, z_1, \zeta_{kN} z_2)$ .

The group  $\Gamma_k$  acts on the moduli space  $\mathcal{P}_{\underline{d}}$  of parabolic sheaves trivialized at infinity via its action on  $\mathbf{S}$  and the *trivial* action on the trivialization at infinity. The fixed point variety  $\mathcal{P}_{\underline{d}}^{\Gamma_k}$  has various connected components, and we are going to describe them in quiver terms. To this end note that  $\Gamma_{kN}$  acts on the moduli space  $\mathcal{P}'_{\underline{d}} = \mathfrak{M}_{\underline{d}}$  of torsion free sheaves on  $\mathcal{S}'$  framed at  $\ell_\infty$  via its action on  $\mathcal{S}'$  and the *trivial* action on the framing. We have  $\mathcal{P}_{\underline{d}}^{\Gamma_k} = \mathfrak{M}_{\underline{d}}^{\Gamma_{kN}}$ . According to Section 2.6,  $\Gamma_{kN}$  acts on  $\mathfrak{M}_{\underline{d}}$  as follows:  $\zeta_{kN}(A_\bullet, B_\bullet, p_\bullet, q_\bullet) = (\zeta_{kN} A_\bullet, \zeta_{kN} B_\bullet, \zeta_{kN} p_\bullet, \zeta_{kN} q_\bullet)$ . To formulate the conclusion we consider the representations of the following quiver  $Q^k$ :



Here the lower indices of  $V$  run through  $\mathbb{Z}/N\mathbb{Z}$ , while the upper indices run through  $\mathbb{Z}/k\mathbb{Z}$ . The relations are as follows:  $0 = A_1 B_N - B_N A_0 + p_1 q_0 : V_N^{-1} \rightarrow V_1^0$ ;  $0 = A_{l+1} B_l - B_l A_l + p_{l+1} q_l : V_l^0 \rightarrow V_{l+1}^0$  for  $1 \leq l \leq N-1$ , and  $0 = A_1 B_N - B_N A_0 : V_N^r \rightarrow V_1^{r+1}$  for  $r \neq -1$ , and  $0 = A_{l+1} B_l - B_l A_l : V_l^r \rightarrow V_{l+1}^r$  in the remaining cases.

We set  $d_l^r := \dim(V_l^r)$ , and we denote by  $\tilde{\underline{d}}$  the collection of positive integers  $(d_l^r)_{l \in \mathbb{Z}/N\mathbb{Z}}^{r \in \mathbb{Z}/k\mathbb{Z}}$ . We set  $\underline{d}(\tilde{\underline{d}}) := (d_1, \dots, d_N)$  where  $d_l = \sum_{r \in \mathbb{Z}/k\mathbb{Z}} d_l^r$ . We denote by  $\mathbf{M}_{\tilde{\underline{d}}}$  the scheme of representations of  $Q^k$  of dimension  $\tilde{\underline{d}}$ . We denote by  $G_{\tilde{\underline{d}}}$  the group  $\prod_{l \in \mathbb{Z}/N\mathbb{Z}}^{r \in \mathbb{Z}/k\mathbb{Z}} \mathrm{GL}(V_l^r)$ ; it acts naturally on  $\mathbf{M}_{\tilde{\underline{d}}}$ . We denote by  $\mathfrak{Z}_{\tilde{\underline{d}}}$  the categorical quotient  $\mathbf{M}_{\tilde{\underline{d}}}/G_{\tilde{\underline{d}}}$ . Furthermore, we consider an open subscheme  $\mathbf{M}_{\tilde{\underline{d}}}^s \subset \mathbf{M}_{\tilde{\underline{d}}}$  formed by all the *stable* representations of  $Q^k$ , i.e. those  $(A_\bullet, B_\bullet, p_\bullet, q_\bullet) \in \mathbf{M}_{\tilde{\underline{d}}}$  such that there is no proper graded subspace  $V_\bullet' \subset V_\bullet$  stable under  $A_\bullet, B_\bullet$  and containing  $p(W_\bullet)$ . The action of  $G_{\tilde{\underline{d}}}$  on  $\mathbf{M}_{\tilde{\underline{d}}}^s$  is free, and we consider the GIT quotient  $\mathfrak{M}_{\tilde{\underline{d}}} = \mathbf{M}_{\tilde{\underline{d}}}^s/G_{\tilde{\underline{d}}}$ . Note that  $\mathfrak{M}_{\tilde{\underline{d}}}$  is nonempty iff  $d_N^0 \geq d_1^1 \geq d_2^1 \geq \dots \geq d_N^1 \geq d_1^2 \geq d_2^2 \geq \dots \geq d_{N-2}^{-1} \geq d_{N-1}^{-1} \geq d_N^{-1}$ .

The above considerations imply the following

**Proposition 2.9.** *The fixed point variety  $\mathcal{P}_{\tilde{\underline{d}}}^{\Gamma^k}$  is a union of connected components isomorphic to  $\mathfrak{M}_{\tilde{\underline{d}}}$ , over all collections  $\tilde{\underline{d}}$  such that  $\underline{d}(\tilde{\underline{d}}) = \underline{d}$ .*

**2.10. Direct image.** Given  $\underline{d} = (d_1, \dots, d_N)$  we consider  $\tilde{\underline{d}} = \tilde{\underline{d}}(\underline{d})$  such that  $d_l^0 = d_l$  for any  $1 \leq l \leq N$ , and  $d_l^r = d_N$  for any  $1 \leq l \leq N$  and  $r \neq 0$ . Then it is easy to see that

$\mathfrak{M}_{\tilde{\underline{d}}} \simeq \mathfrak{M}_{\underline{d}}$ . In effect, all the maps  $B_l$  except for  $B_N : V_N^{-1} \rightarrow V_1^0$ , and the ones in the 0th row, have to be isomorphisms intertwining the corresponding endomorphisms  $A_l$  and  $A_{l+1}$ .

Geometrically, the isomorphism  $\mathcal{P}_{\tilde{\underline{d}}}^{\Gamma_k} \xrightarrow{\sim} \mathcal{P}_{\underline{d}}$  has the following explanation. We have an evident projection  $\psi : \mathbf{S} \rightarrow \mathbf{S} // \Gamma_k \simeq \mathbf{S}$  (the categorical quotient). A  $\Gamma_k$ -invariant point of  $\mathcal{P}_{\underline{d}}$  is represented by a  $\Gamma_k$ -equivariant torsion free parabolic sheaf  $\mathcal{F}$  on  $\mathbf{S}$ . Then  $\psi_* \mathcal{F}$  carries a fiberwise action of  $\Gamma_k$ , and  $(\psi_* \mathcal{F})^{\Gamma_k}$  is a torsion free parabolic sheaf on  $\mathbf{S}$ , trivialized at infinity. Its class in  $\mathcal{P}_{\underline{d}}$  is the image of  $\mathcal{F}$  under the above isomorphism. For this reason, somewhat abusing notation, we will denote this isomorphism by  $\psi_*^{\Gamma_k}$ .

For an arbitrary  $\tilde{\underline{d}}$ , we consider  $\underline{d} = (d_1, \dots, d_N) := (d_1^0, \dots, d_{N-1}^0, d_N^{-1})$ . Then we still have a morphism  $\psi_*^{\Gamma_k} : \mathcal{P}_{\tilde{\underline{d}}}^{\Gamma_k} \rightarrow \mathcal{P}_{\underline{d}}$  which is not necessarily an isomorphism. Let us describe it in quiver terms:  $\psi_*^{\Gamma_k} : \mathfrak{M}_{\tilde{\underline{d}}} \rightarrow \mathfrak{M}_{\underline{d}}$ . We have  $V'_N := V_N^{-1}$ ,  $V'_l := V_l^0$  for  $1 \leq l \leq N-1$ . Furthermore, we have  $A'_l := A_l$  for  $1 \leq l \leq N$ , and  $B'_{N-1}$  is the composition of all  $B$ 's going from  $V_{N-1}^0$  to  $V_N^0$ , then to  $V_1^1$ , and all the way through to  $V_N^{-1}$ ; while all the other  $B'_l$  coincide with the corresponding  $B_l$ . Finally,  $q'_l$  coincides with the corresponding  $q_l$ , and for  $1 \leq l \leq N-1$  the map  $p'_l$  coincides with the corresponding  $p_l$ ; while  $p'_N$  is the composition of  $p_N$  with all the  $B$ 's going from  $V_N^0$  to  $V_1^1$ , and all the way through to  $V_N^{-1}$ .

**2.11. Defect.** Let us give a geometric explanation of what is so special about the components  $\mathcal{P}_{\tilde{\underline{d}}}^{\Gamma_k}$ ,  $\tilde{\underline{d}} = \tilde{\underline{d}}(\underline{d})$  considered in Section 2.10. Namely, they are the only components of the fixed point variety  $\mathcal{P}_{\underline{d}}^{\Gamma_k}$  which contain the *nonempty* open subset formed by the  $\Gamma_k$ -equivariant *locally free* parabolic sheaves.

For an arbitrary  $\Gamma_k$ -equivariant torsion free parabolic sheaf  $\mathcal{F}_{\bullet}$ , there is a notion of the *saturation*  $\hat{\mathcal{F}}_{\bullet}$  (a locally free parabolic sheaf containing  $\mathcal{F}_{\bullet}$ , such that the quotient has a zero-dimensional support). The global sections of this quotient is a  $\mathbb{Z}/N\mathbb{Z}$ -graded  $\Gamma_k$ -module  $\text{def}(\mathcal{F}_{\bullet})$ , the *defect* of  $\mathcal{F}_{\bullet}$ . The class  $[\text{def}(\mathcal{F}_{\bullet})]$  of  $\text{def}(\mathcal{F}_{\bullet})$  in the  $K$ -group of  $\mathbb{Z}/N\mathbb{Z}$ -graded  $\Gamma_k$ -modules is represented by a collection  $\tilde{\underline{d}}$  of integers. The class  $[\text{def}(\mathcal{F}_{\bullet})]$  may vary throughout a connected component of the fixed point variety  $\mathcal{P}_{\underline{d}}^{\Gamma_k}$ . However, its class  $[\overline{\text{def}}(\mathcal{F}_{\bullet})]$  modulo the subgroup spanned by all the collections of the sort  $\tilde{\underline{d}}(\underline{d}')$ ,  $\underline{d}' \in \mathbb{Z}^{\mathbb{Z}/N\mathbb{Z}}$ , is constant throughout a connected component  $\mathcal{P}_{\underline{d}}^{\Gamma_k}$ . Quite evidently, the class  $[\overline{\text{def}}(\mathcal{F}_{\bullet})]$  for  $\mathcal{F}_{\bullet} \in \mathcal{P}_{\underline{d}}^{\Gamma_k}$ , equals the class of  $\tilde{\underline{d}}$ . In particular, in order to have a locally free parabolic sheaf  $\mathcal{F}_{\bullet}$  (i.e. the one with zero defect) in a component  $\mathcal{P}_{\underline{d}}^{\Gamma_k}$  it is necessary and sufficient that  $\tilde{\underline{d}}$  be of the form  $\tilde{\underline{d}}(\underline{d})$  for some  $\underline{d}$ .

### 3. ZASTAVA FOR BLOWN UP PLANE

**3.1. Dented chainsaw.** We consider the representations of the following *dented chainsaw quiver*  $\hat{Q}$

$$\begin{array}{ccccccccccc}
& & A_{N-2} & & A_{N-1} & & & & A_1 & & A_2 \\
& & \curvearrowright & & \curvearrowright & & & & \curvearrowright & & \curvearrowright \\
\cdots & \xrightarrow{B_{N-3}} & V_{N-2} & \xrightarrow{B_{N-2}} & V_{N-1} & \xrightarrow{B_{N-1}} & V_N & \xrightleftharpoons[e]{\delta} & V_0 & \xrightarrow{B_0} & V_1 & \xrightarrow{B_1} & V_2 & \xrightarrow{B_2} & \cdots \\
& \nearrow p_{N-2} & \downarrow q_{N-2} & \nearrow p_{N-1} & \downarrow q_{N-1} & \nearrow p_N & & & \downarrow q_0 & \nearrow p_1 & \downarrow q_1 & \nearrow p_2 & \downarrow q_2 & \nearrow p_3 & \\
& & W_{N-2} & & W_{N-1} & & & & W_0 & & W_1 & & W_2 & & \cdots
\end{array}$$

with relations  $A_{l+1}B_l - B_lA_l + p_{l+1}q_l = 0$  for any  $1 \leq l \leq N-2$ ;  $A_1B_0 - B_0e\delta + p_1q_0 = 0$ ;  $\delta eB_{N-1} - B_{N-1}A_{N-1} + p_Nq_{N-1} = 0$ . Here  $\dim W_l = 1$ ,  $d_N := \dim V_N = d_0 := \dim V_0$ ,  $\dim V_l = d_l$ ,  $l = 1, \dots, N-1$ . We denote by  $\widehat{\mathbf{M}}_{\underline{d}}$  the scheme of representations of  $\widehat{Q}$ . We denote by  $\widehat{G}_{\underline{d}}$  the group  $\prod_{0 \leq l \leq N} \mathrm{GL}(V_l)$ ; it acts naturally on  $\widehat{\mathbf{M}}_{\underline{d}}$ . Performing the celebrated Crawley-Boevey trick, we identify all the lines  $W_l$  with, say  $W_\infty$ , so that  $W_\infty$  is the source of all  $p_l$ , and the target of all  $q_l$ . We will denote a typical representation of  $\widehat{Q}$  by  $Y$ .

**3.2. Stability conditions.** We consider the enhanced dimension vectors  $\underline{\widehat{d}} := (d_0, d_1, d_2, \dots, d_{N-1}, d_N)$ , and  $\underline{\ddot{d}} := (d_0, d_1, d_2, \dots, d_{N-1}, d_N, 1)$  with one extra coordinate equal to  $\dim W_\infty = 1$ . We consider a vector  $\zeta^\bullet = (\zeta_0, \zeta_1, \dots, \zeta_{N-1}, \zeta_N)$  where  $\zeta_N = -1$ ,  $\zeta_0 = 1$ ,  $\zeta_l = 0$  for  $l = 1, \dots, N-1$ . Also, for  $0 < \varepsilon \ll 1$  we consider  $\zeta^- := \zeta^\bullet - (\varepsilon, \dots, \varepsilon)$ . We set  $\zeta_\infty^- := -\langle \zeta^-, \underline{\widehat{d}} \rangle$ , and  $\zeta_\infty^\bullet := -\langle \zeta^\bullet, \underline{\widehat{d}} \rangle$  where  $\langle \cdot, \cdot \rangle$  stands for the sum of products of coordinates (the standard scalar product). Finally, we set  $\widetilde{\zeta}^- := (\zeta^-, \zeta_\infty^-)$ , and  $\widetilde{\zeta}^\bullet := (\zeta^\bullet, \zeta_\infty^\bullet)$ .

For a nonzero  $\widehat{Q}$ -submodule  $Y' \subset Y$  of enhanced dimension  $\underline{\ddot{d}}'$  (where the last coordinate may be either 1 or 0) we define the slope by

$$\theta^-(Y') := \frac{\langle \widetilde{\zeta}^-, \underline{\ddot{d}}' \rangle}{\langle (1, \dots, 1), \underline{\ddot{d}}' \rangle}, \quad \theta^\bullet(Y') := \frac{\langle \widetilde{\zeta}^\bullet, \underline{\ddot{d}}' \rangle}{\langle (1, \dots, 1), \underline{\ddot{d}}' \rangle}.$$

We say that a  $\widehat{Q}$ -module  $Y$  is  $\zeta^-$ -semistable (resp.  $\zeta^\bullet$ -semistable) if for any nonzero submodule  $Y' \subset Y$  we have  $\theta^-(Y') \leq \theta^-(Y)$  (resp.  $\theta^\bullet(Y') \leq \theta^\bullet(Y)$ ). We say  $Y$  is  $\zeta^-$ -stable (resp.  $\zeta^\bullet$ -stable) if the inequality is strict unless  $Y' = Y$ . Note that  $\zeta^-$ -stability is equivalent to  $\zeta^-$ -semistability.

We define a scheme  $\widehat{\mathfrak{M}}_{\underline{d}}$  as the moduli space of  $\zeta^-$ -semistable (equivalently,  $\zeta^-$ -stable)  $\widehat{Q}$ -modules. By GIT,  $\widehat{\mathfrak{M}}_{\underline{d}}$  is the projective spectrum of the ring of  $\widehat{G}_{\underline{d}}$ -semiinvariants in  $\mathbb{C}[\widehat{\mathbf{M}}_{\underline{d}}]$ . Furthermore, we define a scheme  $\widehat{\mathfrak{Z}}_{\underline{d}}$  as the moduli space of  $S$ -equivalence classes of  $\zeta^\bullet$ -semistable  $\widehat{Q}$ -modules. Since the stability condition  $\zeta^\bullet$  lies on a wall of the chamber containing  $\zeta^-$ , we have a projective morphism  $\pi_{\zeta^\bullet, \zeta^-} : \widehat{\mathfrak{M}}_{\underline{d}} \rightarrow \widehat{\mathfrak{Z}}_{\underline{d}}$ .

**3.3. Parabolic sheaves on blow-up.** We stick to the notations of [21]. Namely,  $\mathbb{P}^2$  is the projective plane with homogeneous coordinates  $[z_0 : z_1 : z_2]$ , and  $\ell_\infty \subset \mathbb{P}^2$  is the line “at infinity” given by the equation  $z_0 = 0$ . Furthermore,  $\widehat{\mathbb{P}}^2$  is the blow-up of  $\mathbb{P}^2$  “at the origin” (given by equations  $z_1 = z_2 = 0$ ). It is the closed subvariety of  $\mathbb{P}^2 \times \mathbb{P}^1$  defined by  $\widehat{\mathbb{P}}^2 = \{([z_0 : z_1 : z_2], [z : w]) : z_1w = z_2z\}$ . We denote by  $E$  the exceptional divisor in  $\widehat{\mathbb{P}}^2$ ;

we denote by  $\ell_0 \subset \widehat{\mathbb{P}}^2$  the proper transform of the line  $z_2 = 0$  in  $\mathbb{P}^2$ ; finally, by an abuse of notation, we denote by  $\ell_\infty \subset \widehat{\mathbb{P}}^2$  the proper transform of the line  $\ell_\infty \subset \mathbb{P}^2$ .

We set  $W := W_1 \oplus W_2 \oplus \dots \oplus W_{N-1} \oplus W_0$ . Given an  $N$ -tuple of nonnegative integers  $\underline{d} = (d_0, \dots, d_{N-1})$  we say that a *parabolic sheaf*  $\mathcal{F}_\bullet$  of degree  $\underline{d}$  is an infinite flag of torsion free coherent sheaves of rank  $N$  on  $\widehat{\mathbb{P}}^2$ :  $\dots \subset \mathcal{F}_{-1} \subset \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots$  such that

- (a)  $\mathcal{F}_{k+N} = \mathcal{F}_k(\ell_0)$  for any  $k \in \mathbb{Z}$ ;
- (b)  $ch_1(\mathcal{F}_k) = k[\ell_0]$  for any  $k \in \mathbb{Z}$ : the first Chern classes are proportional to the fundamental class of  $\ell_0$ ;
- (c)  $ch_2(\mathcal{F}_k) = d_i$  for  $i \equiv k \pmod{N}$ ;
- (d)  $\mathcal{F}_0$  is locally free at  $\ell_\infty$  and trivialized at  $\ell_\infty$ :  $\mathcal{F}_0|_{\ell_\infty} = W \otimes \mathcal{O}_{\ell_\infty}$ ;
- (e) For  $-N \leq k \leq 0$  the sheaf  $\mathcal{F}_k$  is locally free at  $\ell_\infty$ , and the quotient sheaves  $\mathcal{F}_k/\mathcal{F}_{-N}$ ,  $\mathcal{F}_0/\mathcal{F}_k$  (both supported at  $\ell_0 \subset \widehat{\mathbb{P}}^2$ ) are locally free at the point  $\ell_0 \cap \ell_\infty$ ; moreover, the local sections of  $\mathcal{F}_k|_{\ell_\infty}$  are those sections of  $\mathcal{F}_0|_{\ell_\infty} = W \otimes \mathcal{O}_{\ell_\infty}$  which take value in  $W_1 \oplus \dots \oplus W_{k+N} \subset W$  at  $\ell_0 \cap \ell_\infty$ .

One can show that the fine moduli space  $\widehat{\mathcal{P}}_{\underline{d}}$  of degree  $\underline{d}$  parabolic sheaves exists, and is a smooth connected quasiprojective variety of dimension  $2d_0 + \dots + 2d_{N-1}$ .

**Theorem 3.4.** (A. Kuznetsov) *There is an isomorphism  $\Xi: \widehat{\mathcal{M}}_{\underline{d}} \xrightarrow{\sim} \widehat{\mathcal{P}}_{\underline{d}}$ .*

The proof occupies the rest of this Section.

**3.5. Stack  $\widehat{\mathcal{S}}_N$ .** We denote by  $\ell' \subset \widehat{\mathbb{P}}^2$  the proper transform of the line  $z_1 = 0$  in  $\mathbb{P}^2$ . We consider the open subvarieties  $U' := \widehat{\mathbb{P}}^2 - \ell_0 \simeq \ell_0 \times \mathbb{A}^1$  with coordinate  $z = z_1 z_2^{-1}$  along  $\mathbb{A}^1$ , and  $U_0 := \widehat{\mathbb{P}}^2 - \ell' \simeq \ell_0 \times \mathbb{A}^1$  with coordinate  $w = z_2 z_1^{-1}$  along  $\mathbb{A}^1$ . We consider the ramified Galois covering  $\theta: \mathbb{A}^1 \rightarrow \mathbb{A}^1$ ,  $w = s^N$  with Galois group  $\Gamma_N$ . We denote by  $\theta: \widetilde{U}_0 \rightarrow U_0$  the base change of this covering under  $U_0 \rightarrow \mathbb{A}^1$ . The action of  $\Gamma_N$  on  $\theta^{-1}(U_0 \cap U')$  is free, and  $\theta^{-1}(U_0 \cap U')/\Gamma_N = U_0 \cap U'$ . We define a 2-dimensional Deligne-Mumford stack  $\widehat{\mathcal{S}}_N$  as the result of gluing  $U'$  and  $\widetilde{U}_0/\Gamma_N$  over the common open  $U_0 \cap U'$ . Note that  $\widehat{\mathcal{S}}_N$  is equipped with a projection  $\vartheta$  to  $\widehat{\mathbb{P}}^2$  which is an isomorphism off  $\ell_0$ . The line in  $\widehat{\mathcal{S}}_N$  lying over  $\ell_0$  will be denoted by  $\ell \subset \widehat{\mathcal{S}}_N$ ; its automorphism group is  $\Gamma_N$ . Since  $N$  is fixed throughout the Section, we will often omit the lower index  $N$  to simplify the notations.

We also have a smooth morphism  $\pi$  (a  $\mathbb{P}^1$ -bundle) from  $\widehat{\mathcal{S}}_N$  to the 1-dimensional stack  $\mathcal{X}_N$  of Section 2.3 such that  $\pi^{-1}(\infty_{\mathcal{X}}) = \ell'$ , and  $\pi^{-1}(0_{\mathcal{X}}) = \ell$ . A section of  $\pi$  sending  $\mathcal{X}$  to  $\ell_\infty \subset \widehat{\mathcal{S}}$  will be denoted by  $\iota$ . We choose a section  $y_2$  of  $\mathcal{O}_{\mathcal{X}}(1)$  with a simple zero at  $0_{\mathcal{X}}$ , and a section  $y_1$  of  $\mathcal{O}_{\mathcal{X}}(N)$  with a simple zero at  $\infty_{\mathcal{X}}$  (in notations of Section 2.2 and Section 2.3 we have  $y_2^N = y$ ,  $y_1 = x$ ). We keep the same names for the corresponding sections of  $\mathcal{O}_{\widehat{\mathcal{S}}}(\ell)$  and  $\mathcal{O}_{\widehat{\mathcal{S}}}(N\ell)$  constant along the fibers of  $\pi$ . Finally, we choose a section  $\epsilon$  of  $\mathcal{O}_{\widehat{\mathcal{S}}}(\ell_\infty - N\ell) = \mathcal{O}_{\widehat{\mathcal{S}}}(\ell_\infty - \ell')$  with a simple zero at  $E$ .

According to [2], there is a one-to-one correspondence between the (torsion free, framed at  $\ell_\infty$ ) sheaves on  $\widehat{\mathcal{S}}$ , and the (torsion free, framed at  $\ell_\infty$ ) sheaves on  $\widehat{\mathbb{P}}^2$  with parabolic structure along  $\ell_0$ . Thus  $\widehat{\mathcal{P}}_{\underline{d}}$  is the moduli space of torsion free sheaves of degree  $\underline{d}$  on  $\widehat{\mathcal{S}}$  framed at  $\ell_\infty$ . More precisely, the framing at  $\ell_\infty$  is an isomorphism  $\iota^*\mathcal{F} \simeq \mathcal{F}_\infty := W_0 \otimes \mathcal{O}_{\mathcal{X}} \oplus W_1 \otimes \mathcal{O}_{\mathcal{X}}(-1) \oplus \dots \oplus W_{N-1} \otimes \mathcal{O}_{\mathcal{X}}(-N+1)$ . For technical reasons, it will be





where  $v$  and  $v'$  are the evident projections, while the only nontrivial summand of  $\varkappa$  is the canonical element in  $\text{Ext}_{\mathcal{X}}^1(V_0(0), V_0(-N-1))$  (identical on  $V_0$ ).

Now we are ready to describe the extension  $\kappa$  of (3.2). Its summand  $\kappa_1$  in  $\text{Ext}_{\mathcal{S}}^1\left(\bigoplus_{l=0}^N V_l(-\ell_{\infty} + (N-l)\ell), \bigoplus_{l=0}^{N-1} W_l((-l-1)\ell)\right)$ , by the adjointness and projection formula for  $\pi^*$ , is the same as the direct sum of two elements:  $\kappa_1'' \in \text{Ext}_{\mathcal{X}}^1\left(\bigoplus_{l=0}^N V_l(N-l), \bigoplus_{l=0}^{N-1} W_l(-l-1)\right)$ , and  $\kappa_1' \in \text{Ext}_{\mathcal{X}}^1\left(\bigoplus_{l=0}^N V_l(-l), \bigoplus_{l=0}^{N-1} W_l(-l-1)\right)$ . We set  $\kappa_1' = 0$ , while the “matrix elements” of  $\kappa_1''$  are as follows:  $(y_2^m)^* q_{l+m} B_{l+m-1} \dots B_l \in \text{Ext}^1(V_l(N-l), W_{l+m}(-l-m-1)) = H^1(\mathcal{X}, \mathcal{O}(-N-m-1)) \otimes \text{Hom}(V_l, W_{l+m}) = \Gamma(\mathcal{X}, \mathcal{O}(m))^* \otimes \text{Hom}(V_l, W_{l+m})$  for  $m \geq 0$ .

Furthermore, the summand  $\kappa_2$  in  $\text{Ext}_{\mathcal{S}}^1\left(\bigoplus_{l=0}^N V_l(-\ell_{\infty} + (N-l)\ell), V_0((-N-1)\ell) \oplus \bigoplus_{l=1}^{N-1} V_l((-l-1)\ell)\right)$ , by the adjointness and projection formula for  $\pi^*$ , is the same as the direct sum of two elements:  $\kappa_2'' \in \text{Ext}_{\mathcal{X}}^1\left(\bigoplus_{l=0}^N V_l(N-l), V_0(-N-1) \oplus \bigoplus_{l=1}^{N-1} V_l(-l-1)\right)$ , and  $\kappa_2' \in \text{Ext}_{\mathcal{X}}^1\left(\bigoplus_{l=0}^N V_l(-l), V_0(-N-1) \oplus \bigoplus_{l=1}^{N-1} V_l(-l-1)\right)$ . We set  $\kappa_2' := \varkappa$ , while the “matrix elements” of  $\kappa_2''$  are as follows:  $(y_2^{N-l})^* e B_{N-1} \dots B_{l+1} B_l \in \text{Ext}^1(V_l(N-l), V_0(-N-1)) = H^1(\mathcal{X}, \mathcal{O}(-2N+l-1)) \otimes \text{Hom}(V_l, V_0) = \Gamma(\mathcal{X}, \mathcal{O}(N-l))^* \otimes \text{Hom}(V_l, V_0)$ ;  $(y_2^m)^* A_{l+m} B_{l+m-1} \dots B_l \in \text{Ext}^1(V_l(N-l), V_{l+m}(-l-m-1)) = H^1(\mathcal{X}, \mathcal{O}(-N-m-1)) \otimes \text{Hom}(V_l, V_{l+m}) = \Gamma(\mathcal{X}, \mathcal{O}(m))^* \otimes \text{Hom}(V_l, V_{l+m})$  for  $m \geq 0$ .

Finally, we are able to describe the morphisms  $C, D$  of (3.1). We have  $C = C_1 + C_2$  where  $C_1$  is the pullback of

$$\pi^* \gamma(-\ell_{\infty} + N\ell) : V_0(-\ell_{\infty}) \oplus \bigoplus_{l=0}^{N-1} V_l(-\ell_{\infty} + (N-l-1)\ell) \rightarrow \bigoplus_{l=0}^N V_l(-\ell_{\infty} + (N-l)\ell),$$

while  $C_2$  factors through the same named morphism

$$C_2 : V_0(-\ell_{\infty}) \oplus \bigoplus_{l=0}^{N-1} V_l(-\ell_{\infty} + (N-l-1)\ell) \rightarrow V_0((-N-1)\ell) \oplus \bigoplus_{l=1}^{N-1} V_l((-l-1)\ell)$$

corresponding by adjointness and projection formula to  $v$  of (3.5).

We have  $D = D_1 + D_2 + D_3$  where  $D_1$  is the pushforward of

$$\pi^* \gamma' : V_0((-N-1)\ell) \oplus \bigoplus_{l=1}^{N-1} V_l((-l-1)\ell) \rightarrow \bigoplus_{l=1}^N V_l(-l\ell),$$

while  $D_2$  is the pushforward of

$$-\pi^* \beta : \bigoplus_{l=0}^{N-1} W_l((-l-1)\ell) \rightarrow \bigoplus_{l=1}^N V_l(-l\ell).$$

The “matrix elements” of  $\beta \in \text{Hom}(\mathcal{G}_{\infty}, \bigoplus_{l=1}^N V_l(-l))$  are as follows:  $p_{l+1} : W_l(-l-1) \rightarrow V_{l+1}(-l-1)$  (modulo the image of  $\gamma' \circ ?$ ). Finally,  $D_3$  factors through the same named

morphism

$$D_3 : \bigoplus_{l=0}^N V_l(-\ell_\infty + (N-l)\ell) \rightarrow \bigoplus_{l=1}^N V_l(-l\ell)$$

corresponding by adjointness and projection formula to  $v'$  of (3.5).

For the reader's convenience we collect some of the above morphisms in the following figure:

$$\begin{array}{ccc}
 V_0(0) \oplus \bigoplus_{l=0}^{N-1} V_l(N-l-1) & \xrightarrow{\gamma} & \bigoplus_{l=0}^N V_l(N-l) \\
 & \nwarrow \kappa_2'' & \nearrow \kappa_1'' \\
 & \bigoplus_{l=0}^{N-1} W_l(-l-1) & \\
 & \downarrow \beta & \\
 V_0(-N-1) \oplus \bigoplus_{l=1}^{N-1} V_l(-l-1) & \xrightarrow{\gamma'} & \bigoplus_{l=1}^N V_l(-l)
 \end{array} \tag{3.6}$$

Here  $\kappa_1'' \circ \gamma = 0$ , while  $\beta \in \text{Hom}(\mathcal{G}_\infty, \bigoplus_{l=1}^N V_l(-l))$  is only defined modulo the image of  $\gamma' \circ ? : \text{Hom}(\mathcal{G}_\infty, V_0(-N-1) \oplus \bigoplus_{l=1}^{N-1} V_l(-l-1)) \rightarrow \text{Hom}(\mathcal{G}_\infty, \bigoplus_{l=1}^N V_l(-l))$ . Finally,  $\beta \circ \kappa_1'' = \gamma' \circ \kappa_2''$ , and  $\kappa_2'' \circ \gamma = 0$ .

**3.7. Inverse construction.** Conversely, given a torsion free sheaf  $\mathcal{G}$  on  $\widehat{\mathcal{S}}$  with a framing  $i^*\mathcal{G} \simeq \mathcal{G}_\infty$  we have  $H^0(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + l\ell)) = H^2(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + l\ell)) = 0$  for  $0 \leq l \leq N$ , and we set  $V_l := H^1(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + l\ell))$ . Furthermore, we set  $\mathcal{G}'' := \pi_*(\mathcal{G}(-\ell_\infty))$ , and  $\mathcal{G}' := \pi_*\mathcal{G}$ , so that  $V_l = H^1(\mathcal{X}, \mathcal{G}''(l))$ .

Let  $\Delta : \widehat{\mathcal{S}} \hookrightarrow \widehat{\mathcal{S}} \times \widehat{\mathcal{S}}$  stand for the diagonal embedding. We have the following exact triangles in the derived category of coherent sheaves on  $\widehat{\mathcal{S}}$ :

$$\dots \rightarrow \mathcal{O}_{\widehat{\mathcal{S}} \times_{\mathcal{X}} \widehat{\mathcal{S}}}(-\ell_\infty, -\ell_\infty + N\ell) \xrightarrow{\xi} \mathcal{O}_{\widehat{\mathcal{S}} \times_{\mathcal{X}} \widehat{\mathcal{S}}} \rightarrow \Delta_* \mathcal{O}_{\widehat{\mathcal{S}}} \rightarrow \mathcal{O}_{\widehat{\mathcal{S}} \times_{\mathcal{X}} \widehat{\mathcal{S}}}(-\ell_\infty, -\ell_\infty + N\ell)[1] \rightarrow \dots \tag{3.7}$$

$$\dots \rightarrow \mathcal{O}_{\widehat{\mathcal{S}} \times_{\mathcal{X}} \widehat{\mathcal{S}}}(-\ell_\infty, 0) \rightarrow \mathcal{O}_{\widehat{\mathcal{S}} \times_{\mathcal{X}} \widehat{\mathcal{S}}} \xrightarrow{\eta} \mathcal{O}_{\ell_\infty \times_{\mathcal{X}} \widehat{\mathcal{S}}} \xrightarrow{\rho} \mathcal{O}_{\widehat{\mathcal{S}} \times_{\mathcal{X}} \widehat{\mathcal{S}}}(-\ell_\infty, 0)[1] \rightarrow \dots \tag{3.8}$$

It follows that  $\Delta_* \mathcal{O}_{\widehat{\mathcal{S}}}$  is the convolution of the following complex of objects of the derived category of coherent sheaves on  $\widehat{\mathcal{S}}$ :

$$\mathcal{O}_{\widehat{\mathcal{S}} \times_{\mathcal{X}} \widehat{\mathcal{S}}}(-\ell_\infty, -\ell_\infty + N\ell) \xrightarrow{\eta \circ \xi} \mathcal{O}_{\ell_\infty \times_{\mathcal{X}} \widehat{\mathcal{S}}} \xrightarrow{\rho} \mathcal{O}_{\widehat{\mathcal{S}} \times_{\mathcal{X}} \widehat{\mathcal{S}}}(-\ell_\infty, 0)[1] \tag{3.9}$$

Now since  $\mathcal{G} \simeq \text{pr}_{2*}(\Delta_* \mathcal{O}_{\widehat{\mathcal{S}}} \otimes \text{pr}_1^* \mathcal{G})$ , we see that  $\mathcal{G}$  is the convolution of the following complex of objects of the derived coherent category of  $\widehat{\mathcal{S}}$ :

$$\pi^* \mathcal{G}''(-\ell_\infty + N\ell) \xrightarrow{\alpha} \pi^* \mathcal{G}_\infty \xrightarrow{\pi^* \beta} \pi^* \mathcal{G}''[1] \tag{3.10}$$

Here  $\beta$  enters the exact triangle

$$\dots \rightarrow \mathcal{G}'' \rightarrow \mathcal{G}' \rightarrow i^* \mathcal{G} \xrightarrow{\beta} \mathcal{G}''[1] \rightarrow \dots \tag{3.11}$$

while  $\alpha$  by adjointness and projection formula is the same as the direct sum of two morphisms  $\alpha' : \mathcal{G}'' \rightarrow \mathcal{G}_\infty$ , and  $\alpha'' : \mathcal{G}''(N) \rightarrow \mathcal{G}_\infty$ . The condition  $\pi^*\beta \circ \alpha = 0$  implies  $\alpha' = 0$ .

Let now  $\Delta^{\mathcal{X}}$  stand for the diagonal embedding  $\mathcal{X} \hookrightarrow \mathcal{X} \times \mathcal{X}$ . Then we have the exact sequences of coherent sheaves on  $\mathcal{X} \times \mathcal{X}$ :

$$0 \rightarrow \mathcal{O}(0, -N) \oplus \bigoplus_{l=0}^{N-1} \mathcal{O}(l, -l-1) \rightarrow \bigoplus_{l=0}^N \mathcal{O}(l, -l) \rightarrow \Delta_*^{\mathcal{X}} \mathcal{O}_{\mathcal{X}} \rightarrow 0 \quad (3.12)$$

$$0 \rightarrow \mathcal{O}(0, -N-1) \oplus \bigoplus_{l=1}^{N-1} \mathcal{O}(l, -l-1) \rightarrow \bigoplus_{l=1}^N \mathcal{O}(l, -l) \rightarrow \Delta_*^{\mathcal{X}} \mathcal{O}_{\mathcal{X}} \rightarrow 0 \quad (3.13)$$

which yield the resolutions (3.3) and (3.4) for  $\mathcal{G}'' \simeq \mathrm{pr}_{2*}(\Delta_*^{\mathcal{X}} \mathcal{O}_{\mathcal{X}} \otimes \mathrm{pr}_1^* \mathcal{G}'')$ . In particular,  $B_l : V_l \rightarrow V_{l+1}$  is induced by  $y_2 : H^1(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + l\ell)) \rightarrow H^1(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + (l+1)\ell))$ , and  $\delta : V_0 \rightarrow V_N$  is induced by  $y_1 : H^1(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty)) \rightarrow H^1(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + N\ell))$ . In these terms,  $\alpha''$  is nothing else than  $\kappa_1''$  of (3.6).

Furthermore, the morphism  $\alpha''(l-N) : \mathcal{G}''(l) \rightarrow \mathcal{G}_\infty(l-N)$  induces the morphism  $V_l = H^1(\mathcal{X}, \mathcal{G}''(l)) \rightarrow H^1(\mathcal{X}, \mathcal{G}_\infty(l-N)) = W_l \oplus \dots \oplus W_{N-1}$  with components  $q_l, q_{l+1}B_l, \dots, q_{N-1}B_{N-2} \dots B_l$ . The morphism  $\beta(l) : \mathcal{G}_\infty(l) \rightarrow \mathcal{G}''(l)[1]$  induces the morphism  $W_0 \oplus \dots \oplus W_{l-1} = H^0(\mathcal{X}, \mathcal{G}_\infty(l)) \rightarrow H^1(\mathcal{X}, \mathcal{G}''(l)) = V_l$  whose last component is  $p_l : W_{l-1} \rightarrow V_l$ .

The exact triangle  $\dots \rightarrow \mathcal{G}'' \rightarrow \mathcal{G}' \rightarrow \mathcal{G}_\infty \rightarrow \mathcal{G}''[1] \rightarrow \dots$  along with the acyclicity of  $\mathcal{G}_\infty$  yields an isomorphism  $V_0 = H^1(\mathcal{X}, \mathcal{G}'') \simeq H^1(\mathcal{X}, \mathcal{G}')$ . Now  $e : V_N \rightarrow V_0$  is induced by  $\epsilon : V_N = H^1(\mathcal{X}, \mathcal{G}''(N)) = H^1(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + N\ell)) \rightarrow H^1(\widehat{\mathcal{S}}, \mathcal{G}) = H^1(\mathcal{X}, \mathcal{G}') = V_0$ .

Finally, the exact triangle  $\dots \rightarrow \mathcal{G}''(l-N) \rightarrow \mathcal{G}'(l-N) \rightarrow \mathcal{G}_\infty(l-N) \rightarrow \mathcal{G}''(l-N)[1] \rightarrow \dots$  yields the long exact sequence

$$0 \rightarrow H^1(\mathcal{X}, \mathcal{G}''(l-N)) \rightarrow H^1(\mathcal{X}, \mathcal{G}'(l-N)) \rightarrow H^1(\mathcal{X}, \mathcal{G}_\infty(l-N)) \rightarrow H^2(\mathcal{X}, \mathcal{G}''(l-N)) \rightarrow \dots \quad (3.14)$$

A resolution

$$0 \rightarrow \mathcal{G}''(l-N) \rightarrow \mathcal{G}'' \oplus \mathcal{G}''(l) \oplus \mathcal{G}''(l+1) \oplus \dots \oplus \mathcal{G}''(N-1) \rightarrow \mathcal{G}''(l+1) \oplus \dots \oplus \mathcal{G}''(N) \rightarrow 0$$

implies  $H^1(\mathcal{X}, \mathcal{G}''(l-N)) \simeq \mathrm{Ker} \left( V_0 \oplus \bigoplus_{m=l}^{N-1} V_m \rightarrow \bigoplus_{m=l+1}^N V_m \right)$  which, together with (3.14), yields an isomorphism

$$H^1(\mathcal{X}, \mathcal{G}'(l-N)) \simeq \mathrm{Ker} \left( V_0 \oplus \bigoplus_{m=l}^{N-1} V_m \oplus \bigoplus_{m=l}^{N-1} W_m \xrightarrow{\varrho} \bigoplus_{m=l+1}^N V_m \right).$$

Here the “matrix coefficients” of  $\varrho$  are as follows:  $\delta : V_0 \rightarrow V_N$ ;  $B_m : V_m \rightarrow V_{m+1}$ ;  $\mathrm{Id} : V_m \rightarrow V_m$ ;  $p_{m+1} : W_m \rightarrow V_{m+1}$ . In particular, we have a morphism  $\varpi : H^1(\mathcal{X}, \mathcal{G}'(l-N)) \rightarrow V_l$ .

Now the morphism  $\epsilon((l-N)\ell) : \mathcal{G}(-\ell_\infty + l\ell) \rightarrow \mathcal{G}((l-N)\ell)$  gives rise to the morphism  $V_l = H^1(\widehat{\mathcal{S}}, \mathcal{G}(-\ell_\infty + l\ell)) \rightarrow H^1(\widehat{\mathcal{S}}, \mathcal{G}((l-N)\ell)) = H^1(\mathcal{X}, \mathcal{G}'(l-N))$ . Composing it with  $\varpi$  we obtain the morphism  $A_l : V_l \rightarrow V_l$ .

Finally, the equivalence of the conditions of  $\mathcal{F}$  being torsion free and the corresponding quiver data being  $\zeta^-$ -stable is proved similarly to Section 7 of [21].

**3.8. Blowdown.** The blowdown morphism  $\widehat{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$  does not lift to a morphism of the stacks  $\widehat{\mathcal{S}} \rightarrow \mathcal{S}'$  (see Section 2.6). It only gives rise to a correspondence  $\widehat{\mathcal{S}} \xleftarrow{\tilde{\mu}} \mathcal{W} \xrightarrow{\tilde{\nu}} \mathcal{S}'$ . Since both  $\widehat{\mathcal{S}}$  and  $\mathcal{S}'$  are toric stacks, this correspondence can be described in toric terms. Namely,  $\widehat{\mathcal{S}}$  is given by a fan  $\widehat{F}$  formed by the vectors  $(1, 0); (0, 1); (-1, 0); (-N, -N)$  in  $\mathbb{Z}^2$ , while  $\mathcal{S}'$  is given by a fan  $F'$  formed by the vectors  $(1, 0); (0, 1); (-N, -N)$ , and  $\mathcal{W}$  is given by a fan  $\tilde{F}$  formed by the vectors  $(1, 0); (0, 1); (-N, 0); (-N, -N)$ . The evident embedding  $F' \subset \tilde{F}$  corresponds to our  $\tilde{\nu}$ . Since  $\tilde{F}$  is obtained from  $\widehat{F}$  by dilating the vector  $(-1, 0)$ , we obtain the desired morphism  $\tilde{\mu} : \mathcal{W} \rightarrow \widehat{\mathcal{S}}$ . We set  $\Pi := \tilde{\nu}_* \tilde{\mu}^* : D^b \text{Coh}(\widehat{\mathcal{S}}) \rightarrow D^b \text{Coh}(\mathcal{S}')$ . Note that the assumptions of Theorem 4.2(2) of [15] are satisfied in our situation.

Given  $\mathcal{F} = \mathcal{G}(\ell) \in \widehat{\mathcal{P}}_{\underline{d}}$  the complex  $\Pi \mathcal{F}$  is not necessarily a torsion free sheaf: it can have the first cohomology (a torsion sheaf at the origin); it is rather a *perverse coherent sheaf*. Its class is well defined in the Zastava space  $\underline{\mathcal{Z}}_{\underline{d}}$  (see Section 2.1). Thus we obtain the morphism  $\widehat{\mathcal{P}}_{\underline{d}} \rightarrow \underline{\mathcal{Z}}_{\underline{d}}$  which factors through the morphism  $\widehat{\mathcal{Z}}_{\underline{d}} \rightarrow \underline{\mathcal{Z}}_{\underline{d}}$  (since  $\underline{\mathcal{Z}}_{\underline{d}}$  is affine,  $\widehat{\mathcal{Z}}_{\underline{d}}$  is normal, and  $\widehat{\mathcal{P}}_{\underline{d}} \rightarrow \widehat{\mathcal{Z}}_{\underline{d}}$  is proper) to be denoted by  $\Pi$ .

**Conjecture 3.9.** In quiver terms,  $\Pi : \widehat{\mathcal{Z}}_{\underline{d}} \rightarrow \underline{\mathcal{Z}}_{\underline{d}}$  sends  $(V_{\bullet}, A_{\bullet}, B_{\bullet}, \delta, e, p_{\bullet}, q_{\bullet})$  to  $(V'_{\bullet}, A'_{\bullet}, B'_{\bullet}, p'_{\bullet}, q'_{\bullet})$  where  $V'_l := V_l$  for  $l = 1, \dots, N$ , and  $A'_l := A_l$  for  $l = 1, \dots, N-1$ , while  $A'_0 := \delta e$ . Furthermore,  $B'_l := B_l$  for  $l = 1, \dots, N-1$ , while  $B'_0 := B_0 e$ . Furthermore,  $p'_l := p_l$  for  $l = 1, \dots, N$ , and  $q'_l := q_l$  for  $l = 1, \dots, N-1$ , while  $q_0 := q_0 e$ .

#### 4. ZASTAVA FOR KLEINIAN BLOWUP

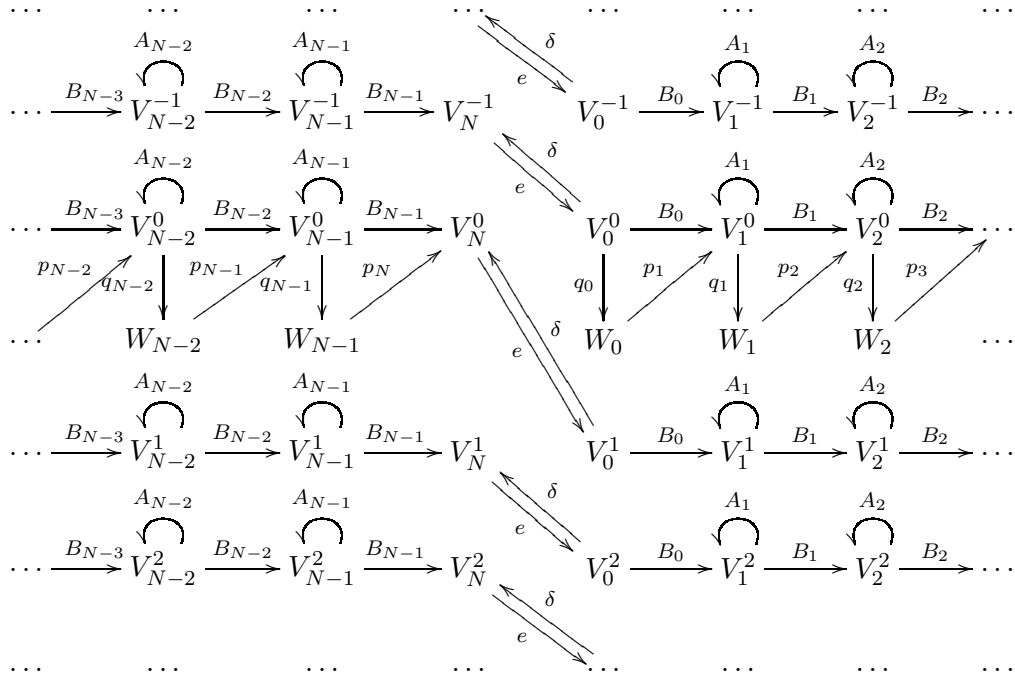
**4.1. Kleinian Blowup.** We consider  $\mathcal{S}'_1 = \mathbb{P}^2$  with homogeneous coordinates  $[z_0 : z_1 : z_2]$ . We blow up the sheaf of ideals  $I$  supported at the origin, and generated locally by  $(z_1^k, z_2)$ . This blowup is a singular toric surface lying in the product  $\mathbb{P}^2 \times \mathbb{P}^1$  and defined by equation  $z_1^k w = z_2 z$  where  $[z : w]$  are the homogeneous coordinates on  $\mathbb{P}^1$ . In fact, it has a unique singular point, lying in a chart  $\overline{U}^2$  with coordinates  $z_1, z_2, z$  satisfying  $z_2 z = z_1^k$ . We define a smooth toric stack  $\widehat{\mathcal{S}}_1^k$  as the stacky resolution of our blowup at the singular point. The neighbourhood  $U^2$  of the stacky point (the preimage of  $\overline{U}^2$ ) is isomorphic to  $\mathbb{A}^2/\Gamma_k$  (with hyperbolic action). This stack is given by a fan  $\widehat{F}^k$  formed by the vectors  $(1, 0); (0, 1); (-1, k-1); (-1, -1)$  in  $\mathbb{Z}^2$ .

Let  $\ell_0 \subset \widehat{\mathcal{S}}_1^k$  be the proper transform of the line  $\ell_0 \subset \mathcal{S}'_1$  given by the equation  $z_2 = 0$ . It lies in the union of two charts  $U^1$  with coordinates  $z_1, w$  and  $U^\infty$  with coordinates  $z_0, z_2$ . We consider the ramified Galois coverings  $U_N^1$  with coordinates  $z_1, \sqrt[N]{w}$ , and  $U_N^\infty$  with coordinates  $z_0, \sqrt[N]{z_2}$ , with Galois group  $\Gamma_N$ . Gluing the stacky quotients  $U_N^1/\Gamma_N$ , and  $U_N^\infty/\Gamma_N$  with  $U^2$ , we obtain the smooth toric stack  $\widehat{\mathcal{S}}_N^k$ . It is given by a fan  $\widehat{F}_N^k$  formed by the vectors  $(1, 0); (0, 1); (-1, k-1); (-N, -N)$  in  $\mathbb{Z}^2$ . The preimage of  $\ell_0 \subset \widehat{\mathcal{S}}_1^k$  is denoted by  $\ell \subset \widehat{\mathcal{S}}_N^k$ . Its automorphism group is  $\Gamma_N$ .

We have a correspondence  $\widehat{\mathcal{S}}_N^k \xleftarrow{\tilde{\mu}_k} \mathcal{W}^k \xrightarrow{\tilde{\nu}^k} \mathcal{S}'_N$  (cf. Section 3.8) where  $\mathcal{W}^k$  is given by a fan  $\tilde{F}^k$  formed by the vectors  $(1, 0); (0, 1); (-N, N(k-1)); (-N, -N)$ . The evident embedding  $F' \subset \tilde{F}^k$  corresponds to our  $\tilde{\nu}^k$ . Since  $\tilde{F}^k$  is obtained from  $\widehat{F}_N^k$  by dilating the vector  $(-1, k-1)$ , we obtain the desired morphism  $\tilde{\mu}_k : \mathcal{W}^k \rightarrow \widehat{\mathcal{S}}_N^k$ .

**4.2. Parabolic sheaves on Kleinian Blowup.** Let  $\widehat{\mathcal{P}}^k$  be the moduli space of rank  $N$  torsion free parabolic sheaves on  $\widehat{\mathcal{S}}_1^k$  (with parabolic structure along  $\ell_0$ , and with trivial first Chern class) trivialized at infinity; equivalently,  $\widehat{\mathcal{P}}^k$  is the moduli space of rank  $N$  torsion free sheaves on  $\widehat{\mathcal{S}}_N^k$  framed at  $\ell_\infty : \iota^* \mathcal{F} \simeq \mathcal{F}_\infty := W_0 \otimes \mathcal{O}_X \oplus W_1 \otimes \mathcal{O}_X(-1) \oplus \dots \oplus W_{N-1} \otimes \mathcal{O}_X(N-1)$  (and with first Chern class trivial off  $\ell_\infty$ ). Given  $\mathcal{F} \in \widehat{\mathcal{P}}^k$  the complex  $\Pi^k \mathcal{F} := \tilde{\nu}_*^k \tilde{\mu}_k^* \mathcal{F}$  is not necessarily a torsion free sheaf on  $\mathcal{S}'_N$ : it can have the first cohomology (a torsion sheaf at the origin); it is rather a perverse coherent sheaf. Its class is well defined in the Zastava space  $\mathfrak{Z}$ . Thus we obtain the morphism  $\widehat{\mathcal{P}}^k \rightarrow \mathfrak{Z}$ . Our goal is to describe the moduli space  $\widehat{\mathcal{P}}^k$  (in particular, to number its connected components) in quiver terms, as well as the morphism  $\widehat{\mathcal{P}}^k \rightarrow \mathfrak{Z}$ . A connected component of  $\widehat{\mathcal{P}}^k$  will be called *good* if it contains a *nonempty* open subset formed by *locally free* parabolic sheaves.

**4.3. Rift.** We consider the representations of the following *rift* quiver  $\widehat{Q}^k$ :



Here the upper indices of  $V$  run through  $\mathbb{Z}/k\mathbb{Z}$ . The dimension of  $V_l^r$  is denoted by  $d_l^r$ . We consider the dimension vector  $\widehat{\underline{d}} := (d_l^r)_{0 \leq l \leq N, r \in \mathbb{Z}/k\mathbb{Z}}$ . Furthermore,  $\dim W_l = 1$ , and all these lines are identified with, say  $W_\infty$ , so that  $W_\infty$  is the source of all  $p_l$  and the target of all  $q_l$ .

Relations:  $0 = \delta e B_{N-1} - B_{N-1} A_{N-1} + p_N q_{N-1} : V_{N-1}^0 \rightarrow V_N^0$ .

$0 = A_1 B_0 - B_0 e \delta + p_1 q_0 : V_0^0 \rightarrow V_1^0$ .

$0 = A_{l+1} B_l - B_l A_l + p_{l+1} q_l : V_l^0 \rightarrow V_{l+1}^0$  for  $l = 1, \dots, N-2$ .

$0 = \delta e B_{N-1} - B_{N-1} A_{N-1} : V_{N-1}^r \rightarrow V_N^r$  for  $r \neq 0$ .

$0 = A_1 B_0 - B_0 e \delta : V_0^r \rightarrow V_1^r$  for  $r \neq 0$ .

$0 = A_{l+1} B_l - B_l A_l : V_l^r \rightarrow V_{l+1}^r$  for  $l = 1, \dots, N-2, r \neq 0$ .

We denote a typical representation of  $\widehat{Q}^k$  by  $Y$ . We denote by  $M_{\underline{d}}$  the scheme of representations of  $\widehat{Q}^k$  of dimension  $\underline{d}$ . We denote by  $G_{\underline{d}}$  the group  $\prod_{0 \leq l \leq N}^{r \in \mathbb{Z}/k\mathbb{Z}} \mathrm{GL}(V_l^r)$ ; it acts naturally on  $M_{\underline{d}}$ .

**4.4. Stability conditions.** We consider the enhanced dimension vector  $\ddot{d} := (d_l^r, 1)$  with one extra coordinate equal to  $\dim W_\infty = 1$ . We consider a vector  $\zeta^\bullet = (\zeta_l^r)$  where  $\zeta_N^r = -1$ ,  $\zeta_0^r = 1$ ,  $\zeta_l^r = 0$  for  $1 \leq l \leq N-1$ . Also, for  $0 < \varepsilon \ll 1$  we consider  $\zeta^- := \zeta^\bullet - (\varepsilon, \dots, \varepsilon)$ . We set  $\zeta_\infty^- := -\langle \zeta^-, \widehat{d} \rangle$ , and  $\zeta_\infty^\bullet := -\langle \zeta^\bullet, \widehat{d} \rangle$  where  $\langle \cdot, \cdot \rangle$  stands for the sum of products of coordinates (the standard scalar product). Finally, we set  $\tilde{\zeta}^- := (\zeta^-, \zeta_\infty^-)$ , and  $\tilde{\zeta}^\bullet := (\zeta^\bullet, \zeta_\infty^\bullet)$ .

For a nonzero  $\widehat{Q}^k$ -submodule  $Y' \subset Y$  of enhanced dimension  $\ddot{d}'$  (where the last coordinate may be either 1 or 0) we define the slope by

$$\theta^-(Y') := \frac{\langle \tilde{\zeta}^-, \ddot{d}' \rangle}{\langle (1, \dots, 1), \ddot{d}' \rangle}, \quad \theta^\bullet(Y') := \frac{\langle \tilde{\zeta}^\bullet, \ddot{d}' \rangle}{\langle (1, \dots, 1), \ddot{d}' \rangle}.$$

We say that a  $\widehat{Q}^k$ -module  $Y$  is  $\zeta^-$ -semistable (resp.  $\zeta^\bullet$ -semistable) if for any nonzero submodule  $Y' \subset Y$  we have  $\theta^-(Y') \leq \theta^-(Y)$  (resp.  $\theta^\bullet(Y') \leq \theta^\bullet(Y)$ ). We say  $Y$  is  $\zeta^-$ -stable (resp.  $\zeta^\bullet$ -stable) if the inequality is strict unless  $Y' = Y$ . Note that  $\zeta^-$ -stability is equivalent to  $\zeta^-$ -semistability.

We define a scheme  $\widehat{\mathfrak{M}}_{\underline{d}}^k$  as the moduli space of  $\zeta^-$ -semistable (equivalently,  $\zeta^-$ -stable)  $\widehat{Q}^k$ -modules. By GIT,  $\widehat{\mathfrak{M}}_{\underline{d}}^k$  is the projective spectrum of the ring of  $\widehat{G}_{\underline{d}}$ -semiinvariants in  $\mathbb{C}[\widehat{M}_{\underline{d}}]$ . Furthermore, we define a scheme  $\widehat{\mathfrak{Z}}_{\underline{d}}^k$  as the moduli space of  $S$ -equivalence classes of  $\zeta^\bullet$ -semistable  $\widehat{Q}^k$ -modules. Since the stability condition  $\zeta^\bullet$  lies on a wall of the chamber containing  $\zeta^-$ , we have a projective morphism  $\pi_{\zeta^\bullet, \zeta^-} : \widehat{\mathfrak{M}}_{\underline{d}}^k \rightarrow \widehat{\mathfrak{Z}}_{\underline{d}}^k$ .

**Theorem 4.5.** *A good connected component of  $\widehat{\mathcal{P}}^k$  is isomorphic to  $\widehat{\mathfrak{M}}_{\underline{d}}$  for a dimension vector  $\underline{d}$  such that  $d_0^0 = d_N^0$ , and for  $r \neq 0$ ,  $d_l^r = d_m^r \forall l, m = 1, \dots, N$ .*

The proof is given in the next Subsection.

**4.6. The action of  $\Gamma_k$ .** The action of  $\Gamma_k$  on  $S'_1$  (see Section 2.8) lifts to the action of  $\Gamma_k$  on the blowup  $\widehat{S}_1$ , and also lifts to the action of  $\Gamma_{kN}$  on  $\widehat{S}_N$ . Hence  $\Gamma_k$  acts on the moduli space  $\widehat{\mathcal{P}}_{\underline{d}}$  of parabolic sheaves on  $\widehat{S}_1$  trivialized at infinity via its action on  $\widehat{S}_1$  and the *trivial* action on the trivialization at infinity.

The fixed point variety  $\widehat{\mathcal{P}}_{\underline{d}}^{\Gamma_k} = \widehat{\mathfrak{M}}_{\underline{d}}^{\Gamma_{kN}}$  can be described in quiver terms as well. Namely, the construction of quiver in Section 3.7 implies that the action of the generator  $\zeta_{kN}$  of  $\Gamma_{kN}$  on the quiver components works as follows:  $\zeta_{kN}(A_\bullet, B_\bullet, e, \delta, p_\bullet, q_\bullet) = (A_\bullet, \zeta_{kN} B_\bullet, \zeta_k^{-2} e, \zeta_k^2 \delta, \zeta_{kN} p_\bullet, q_\bullet)$ . It follows that the various connected components of  $\widehat{\mathfrak{M}}_{\underline{d}}^{\Gamma_{kN}}$  are isomorphic to  $\widehat{\mathfrak{M}}_{\underline{d}}^k$  for various dimension vectors  $\underline{d}$  such that  $\underline{d} = (\sum_{r \in \mathbb{Z}/k\mathbb{Z}} d_0^r, \sum_{r \in \mathbb{Z}/k\mathbb{Z}} d_1^r, \dots, \sum_{r \in \mathbb{Z}/k\mathbb{Z}} d_N^r)$ .

Among these connected components we single out the ones classifying the parabolic sheaves with  $\Gamma_k$ -equivariantly trivial determinant. This is the condition  $d_0^r = d_N^r$  for any

$r \in \mathbb{Z}/k\mathbb{Z}$ . Also, we single out the components classifying the  $\Gamma_k$ -equivariant parabolic sheaves with the *trivial* defect class  $[\overline{\text{def}}(\mathcal{F}_\bullet)]$  (see Section 2.11). This is the condition that for  $r \neq 0$ ,  $d_l^r = d_m^r \forall l, m = 1, \dots, N$ . We will refer to such connected components (satisfying both of the above conditions) as the *admissible* ones.

Now let us consider the stacks  $\widehat{\mathcal{S}}_1/\Gamma_k$ , and  $\widehat{\mathcal{S}}_1/|\Gamma_k$ . The latter stands for the coarse (categorical) quotient in a neighbourhood of  $\ell_0$  (but the stacky quotient elsewhere). We have an evident projection  $\phi : \widehat{\mathcal{S}}_1/\Gamma_k \rightarrow \widehat{\mathcal{S}}_1/|\Gamma_k$ . Given a  $\Gamma_k$ -equivariant parabolic sheaf  $\mathcal{F}$  on  $\widehat{\mathcal{S}}_1$  lying in an admissible connected component  $\widehat{\mathfrak{M}}_{\underline{d}}^k$ , the parabolic sheaf  $\phi_*\mathcal{F}$  is trivialized at infinity. Similarly to Section 2.10,  $\phi_*$  induces an isomorphism of an admissible connected component  $\widehat{\mathfrak{M}}_{\underline{d}}^k$  with a connected component of the moduli space of torsion free parabolic sheaves on  $\widehat{\mathcal{S}}_1/|\Gamma_k$  (with the trivial action of  $\Gamma_k$  at the trivialization at infinity). However, the stacks  $\widehat{\mathcal{S}}_1/|\Gamma_k$  and  $\widehat{\mathcal{S}}_1^k$  are isomorphic off infinity, so the latter connected component is nothing else than a connected component of  $\widehat{\mathcal{P}}^k$ .

This completes the proof of Theorem 4.5.

**4.7. The morphism  $\Pi^k$ .** For an admissible dimension vector  $\widehat{\underline{d}}$  we denote the corresponding connected component of  $\widehat{\mathcal{P}}^k$  by  $\widehat{\mathcal{P}}_{\underline{d}}^k \simeq \widehat{\mathfrak{M}}_{\underline{d}}^k$ . We are going to describe in quiver terms the morphism  $\widehat{\mathcal{P}}_{\underline{d}}^k \rightarrow \mathfrak{Z}$  of Section 4.2. Note that since  $\mathfrak{Z}$  is affine,  $\pi_{\zeta^\bullet, \zeta^-} : \widehat{\mathfrak{M}}_{\underline{d}}^k \rightarrow \widehat{\mathfrak{Z}}_{\underline{d}}^k$  is proper, and  $\widehat{\mathfrak{Z}}_{\underline{d}}^k$  is normal, the morphism  $\widehat{\mathcal{P}}_{\underline{d}}^k \rightarrow \mathfrak{Z}$  factors as the composition of  $\pi_{\zeta^\bullet, \zeta^-}$  and a certain morphism  $\widehat{\mathfrak{Z}}_{\underline{d}}^k \rightarrow \mathfrak{Z}$  to be denoted by  $\Pi^k$ .

The comparison of constructions of Section 4.6, Section 2.10, Section 3.8, and Conjecture 3.9 implies that  $\Pi^k = \psi_*^{\Gamma_k} \circ \Psi^k$  where  $\Psi^k : \widehat{\mathfrak{Z}}_{\underline{d}}^k \rightarrow \mathfrak{M}_{\underline{d}}^k$  is defined as follows. First,  $\underline{\widetilde{d}} := (d_l^r)_{1 \leq l \leq N}^{r \in \mathbb{Z}/k\mathbb{Z}}$  is obtained from the vector  $\widehat{\underline{d}}$  just by erasing the coordinates  $d_0^r$ ,  $r \in \mathbb{Z}/k\mathbb{Z}$ . Second,  $\Psi^k$  acts on the quiver data as follows:  $'V_l^r := V_l^r$  for  $1 \leq l \leq N$ ,  $r \in \mathbb{Z}/k\mathbb{Z}$ . Furthermore,  $'B_N := B_0 e : V_N^r \rightarrow V_1^{r+1}$  and  $'A_N := \delta e : V_N^r \rightarrow V_N^r$  for  $r \in \mathbb{Z}/k\mathbb{Z}$ . Furthermore,  $'q_0 := q_0 e : V_N^{-1} \rightarrow W_0$ , and all the other primed letters are equal to the corresponding letters without primes.

In particular, we see that  $\Pi^k(\widehat{\mathfrak{Z}}_{\underline{d}}^k)$  lands into the connected component  $\mathfrak{Z}_{\underline{d}}$  where  $\underline{d} = (d_1^0, \dots, d_N^0)$ .

**4.8. An open piece.** We consider the following (admissible) dimension vector:  $d_l^r = v_r$  for any  $l = 0, \dots, N$ . Let  $\mathfrak{U}_{\underline{d}}^k \subset \widehat{\mathfrak{M}}_{\underline{d}}^k$  be the open subset given by the condition that  $B_{N-1}B_{N-2} \dots B_1B_0 : V_0^r \rightarrow V_N^r$  is an isomorphism for any  $r \in \mathbb{Z}/k\mathbb{Z}$  (equivalently, all the  $B_l$  are isomorphisms). This open subset classifies the parabolic sheaves on  $\widehat{\mathcal{S}}_1^k$  *trivial* on  $\ell_0$ . Then the trivialization at infinity extends through  $\ell_0$  as well, and we are left with a torsion free sheaf on the open set  $U^2$  (notations of Section 4.1) trivialized at infinity. Such moduli space is the classical Nakajima quiver variety  $\mathfrak{M}(v, w)$  of type  $\widetilde{A}_{k-1}$  where  $w = (N, 0, \dots, 0)$ , and  $v = (v_0, \dots, v_{k-1})$ .

The isomorphism  $\Phi : \mathfrak{U}_{\underline{d}}^k \xrightarrow{\sim} \mathfrak{M}(v, w)$  in quiver terms is given by  $W_0'' := W_0 \oplus \dots \oplus W_{N-1}$ , and  $W_r'' := 0$  for  $r \neq 0$ . Furthermore,  $V_r'' := V_0^r$ . Furthermore,  $B_r'' := eB_{N-1}B_{N-2} \dots B_1B_0 :$

$V_r'' \rightarrow V_{r+1}''$ , while  $A_r'' := (B_{N-1}B_{N-2}\dots B_1B_0)^{-1}\delta : V_r'' \rightarrow V_{r-1}''$ . Finally,  $p_0'' := \bigoplus_{1 \leq l \leq N} p^l : W_0'' \rightarrow V_0''$ , and  $q_0'' := \bigoplus_{0 \leq l \leq N-1} q^l : V_0'' \rightarrow W_0''$ , where  $p^l := B_0^{-1}B_1^{-1}\dots B_{l-2}^{-1}B_{l-1}^{-1}p_l : W_l \rightarrow V_0^0$ , and  $q^l := q_lB_{l-1}B_{l-2}\dots B_1B_0 : V_0^0 \rightarrow W_l$ .

**4.9. The image of  $\Pi^k$ .** Note that while Theorem 4.5 provides the necessary (admissibility) conditions for a component  $\widehat{\mathcal{P}}_{\underline{d}}^k$  to be good, it does not give the sufficient conditions. Let us give such sufficient conditions in the setup of Section 4.8. Thus we restrict ourselves to the components  $\widehat{\mathcal{M}}_{\underline{d}}^k$  which contain the *nonempty* open subset formed by the *locally free* parabolic sheaves. Equivalently, we are interested in the components  $\mathcal{U}_{\underline{d}}^k \simeq \mathfrak{M}(v, w)$  which contain the *nonempty* open subset formed by the vector bundles on  $U^2$ . The corresponding dimension vectors  $(d_i^v)$  (equivalently, vectors  $v$ ), will be called *good*. The well-known Nakajima criterion states that  $v$  is good iff the  $\widehat{\mathfrak{sl}(k)}$ -weight  $w - Cv$  is dominant and has nonzero multiplicity in the level  $N$  vacuum integrable module  $L(w)$ . Here  $C$  is the affine Cartan matrix of  $\tilde{A}_{k-1}$ . More explicitly, the dominance condition reads as follows:  $v_0 + v_2 \geq 2v_1, \dots, v_{k-2} + v_0 \geq 2v_{k-1}, v_{k-1} + v_1 + N \geq 2v_0$ . In particular,  $v_0 \geq v_i \ \forall i \in \mathbb{Z}/k\mathbb{Z}$ .

Let us note that the type of our bundles on  $U^2$  at the hyperbolic point is  $\bar{\lambda} = {}^t(w - Cv)$  where the transposition is extensively discussed in [4], [5]. In notations of *loc. cit.* (especially Section 7 of [4]), we have  $\mathfrak{M}^{\text{reg}}(v, w) \simeq \text{Bun}_{\text{SL}(N), \mu}^{\lambda}(\mathbb{A}^2/\Gamma_k)$  where  $\lambda, \mu$  are the following integrable  $\mathfrak{sl}(N)_{\text{aff}}$ -weights:  $\lambda = (k, \bar{\lambda}, \frac{a + \frac{(\bar{\mu}, \bar{\mu})}{2} - \frac{(\bar{\lambda}, \bar{\lambda})}{2}}{k})$ ,  $\mu = (k, 0, 0)$ . Here  $a$  stands for the second Chern class of the  $\Gamma_k$ -equivariant vector bundles on a compactification of  $\mathbb{A}^2$ . According to [4], the Nakajima criterion can be equivalently reformulated as follows:  $v$  is good iff  $\mu$  has nonzero multiplicity in the level  $k$  integrable  $\mathfrak{sl}(N)_{\text{aff}}$ -module  $L(\lambda)$ . In particular,  $\lambda \geq \mu$ , i.e. the difference  $\alpha := \lambda - \mu$  is a linear combination of simple roots of  $\mathfrak{sl}(N)_{\text{aff}}$  with coefficients in  $\mathbb{N}$ .

We view  $\alpha$  as a vector with coordinates  $(\alpha_1, \dots, \alpha_N)$ . It is easy to see that  $(\alpha_1, \dots, \alpha_N) \leq (v_0, \dots, v_0)$  componentwise, and hence we have an embedding  $\mathfrak{Z}_{\alpha} \hookrightarrow \mathfrak{Z}_{(v_0, \dots, v_0)}$  adding the defect of the complementary degree at the origin.

From now on let us write  $\widehat{\mathfrak{Z}}_{\mu}^{\lambda}$  for  $\widehat{\mathfrak{Z}}_{\underline{d}}^k$ .

**Conjecture 4.10.** Consider a good component  $\widehat{\mathfrak{Z}}_{\mu}^{\lambda} = \widehat{\mathfrak{Z}}_{\underline{d}}^k \xrightarrow{\Pi^k} \mathfrak{Z}_{v_0, \dots, v_0}$ .

(a) The image of  $\Pi^k$  is contained in  $\mathfrak{Z}_{\alpha} \subset \mathfrak{Z}_{v_0, \dots, v_0}$ , so we may and will view  $\Pi^k$  as a morphism  $\widehat{\mathfrak{Z}}_{\mu}^{\lambda} \rightarrow \mathfrak{Z}_{\alpha}$ .

(b) The morphism  $\Pi^k : \widehat{\mathfrak{Z}}_{\mu}^{\lambda} \rightarrow \mathfrak{Z}_{\alpha}$  is birational and stratified semismall, so that the direct image  $\Pi_*^k \text{IC}(\widehat{\mathfrak{Z}}_{\mu}^{\lambda})$  is a direct sum of IC-sheaves of certain strata of  $\mathfrak{Z}_{\alpha}$  with certain multiplicities.

(c) For  $\beta \leq \alpha$ , and the corresponding stratum  $\mathfrak{Z}_{\beta} \subset \mathfrak{Z}_{\alpha}$ , the multiplicity  $m_{\beta}$  of  $\text{IC}(\mathfrak{Z}_{\beta})$  in  $\Pi_*^k \text{IC}(\widehat{\mathfrak{Z}}_{\mu}^{\lambda})$  equals the weight multiplicity  $L^{\lambda}(\lambda - \alpha + \beta)$  of the integrable  $\mathfrak{sl}(N)_{\text{aff}}$ -module  $L^{\lambda}$ .

## 5. GENERAL $G$



**5.1. Arbitrary groups.** Let  $G$  be an almost simple simply connected group with the Lie algebra  $\mathfrak{g}$ . We have the adjoint representation  $G \rightarrow \mathrm{SL}(\mathfrak{g}) = \mathrm{SL}(N)$  where  $N = \dim \mathfrak{g}$ . We choose a Borel subgroup  $B_N$  of  $\mathrm{SL}(N)$  containing the image of the positive Borel subgroup  $B \subset G$ .

We consider the moduli space of  $G$ -bundles on  $\widehat{\mathcal{S}}_1/\Gamma_k$  (see Section 4.6) equipped with a reduction to  $B$  along  $\ell_0$  and framing at  $\ell_\infty$ . The component of this moduli space having an open piece  $\mathrm{Bun}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k)$  (cf. Section 4.9) will be denoted by  $\widetilde{Z}_{G,\mu}^\lambda$ . Here  $\lambda, \mu \in \Lambda_{\mathrm{aff},k}^+$ ,  $\lambda \geq \mu = (k, 0, 0)$ . The adjoint homomorphism  $\mathrm{ad} : (G, B) \rightarrow (\mathrm{SL}(N), B_N)$  induces a closed embedding  $\mathrm{ad} : \widetilde{Z}_{G,\mu}^\lambda \hookrightarrow \widetilde{Z}_{\mathrm{SL}(N), \mathrm{ad}_* \mu}^{\mathrm{ad}_* \lambda}$ . We define  $\overline{\mathcal{G}Z}_{G_{\mathrm{aff}}}^{\lambda, \alpha}$  as the closure of  $\mathrm{ad}(\widetilde{Z}_{G,\mu}^\lambda)$  in  $\widehat{\mathcal{Z}}_{\mathrm{ad}_* \mu}^{\mathrm{ad}_* \lambda} \supset \widetilde{\mathcal{Z}}_{\mathrm{ad}_* \mu}^{\mathrm{ad}_* \lambda}$ ; here  $\alpha := \lambda - \mu$ . Then  $\Pi^k$  restricts to the proper morphism  $\phi : \overline{\mathcal{G}Z}_{G_{\mathrm{aff}}}^{\lambda, \alpha} \rightarrow Z_{G_{\mathrm{aff}}}^\alpha$ .

**Conjecture 5.2.** (a) *The morphism  $\phi : \overline{\mathcal{G}Z}_{G_{\mathrm{aff}}}^{\lambda, \alpha} \rightarrow Z_{G_{\mathrm{aff}}}^\alpha$  is birational and stratified semismall, so that the direct image  $\phi_* \mathrm{IC}(\overline{\mathcal{G}Z}_{G_{\mathrm{aff}}}^{\lambda, \alpha})$  is a direct sum of IC-sheaves of certain strata of  $Z_{G_{\mathrm{aff}}}^\alpha$  with certain multiplicities.*

(b) *For  $\beta \leq \alpha$ , and the corresponding stratum  $Z_{G_{\mathrm{aff}}}^\beta \subset Z_{G_{\mathrm{aff}}}^\alpha$ , the multiplicity  $m_\beta$  of  $\mathrm{IC}(Z_{G_{\mathrm{aff}}}^\beta)$  in  $\phi_* \mathrm{IC}(\overline{\mathcal{G}Z}_{G_{\mathrm{aff}}}^{\lambda, \alpha})$  equals the weight multiplicity  $L^\lambda(\lambda - \alpha + \beta)$  of the integrable  $G_{\mathrm{aff}}^\vee$ -module  $L^\lambda$ .*

**5.3. Repellents.** We have an intermediate open subset  $\widetilde{Z}_{\mathrm{SL}(N), \mathrm{ad}_* \mu}^{\mathrm{ad}_* \lambda} \subset \check{Z}_{\mathrm{SL}(N), \mathrm{ad}_* \mu}^{\mathrm{ad}_* \lambda} \subset \widehat{\mathcal{Z}}_{\mathrm{ad}_* \mu}^{\mathrm{ad}_* \lambda}$  specified in quiver terms by the condition that the composition  $B_{N-1} B_{N-2} \dots B_1 B_0 : V_0^r \rightarrow V_N^r$  is an isomorphism for any  $r \in \mathbb{Z}/k\mathbb{Z}$ , cf. Section 4.8. It is nothing else than the Uhlenbeck space  $\mathcal{U}_{\mathrm{SL}(N), \mathrm{ad}_* \mu}^{\mathrm{ad}_* \lambda}(\mathbb{A}^2/\Gamma_k)$ . The closure of  $\mathrm{ad}(\widetilde{Z}_{G,\mu}^\lambda)$  in  $\mathcal{U}_{\mathrm{SL}(N), \mathrm{ad}_* \mu}^{\mathrm{ad}_* \lambda}(\mathbb{A}^2/\Gamma_k) = \check{Z}_{\mathrm{SL}(N), \mathrm{ad}_* \mu}^{\mathrm{ad}_* \lambda} \supset \widetilde{\mathcal{Z}}_{\mathrm{SL}(N), \mathrm{ad}_* \mu}^{\mathrm{ad}_* \lambda}$  is nothing else than the Uhlenbeck space  $\mathcal{U}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k) = \overline{\mathcal{W}}_{G_{\mathrm{aff},\mu}}^\lambda$ . In Section 3.2 of [6] we have introduced the locally closed subvariety  $\mathfrak{T}_\mu^e \subset \mathcal{U}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k) = \overline{\mathcal{W}}_{G_{\mathrm{aff},\mu}}^\lambda$  as the repellent of a certain  $\mathbb{C}^*$ -action ( $e$  stands for the neutral element of the affine Weyl group). We conjecture that the central fiber  $\phi^{-1}(i_\alpha^\alpha(0)) \cap \mathcal{U}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k)$  coincides with the repellent  $\mathfrak{T}_\mu^e \subset \mathcal{U}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k)$ . We can prove only one inclusion.

**Proposition 5.4.**  $\mathfrak{T}_\mu^e \subset \phi^{-1}(i_\alpha^\alpha(0)) \cap \mathcal{U}_{G,\mu}^\lambda(\mathbb{A}^2/\Gamma_k)$ .

*Proof.* If a point  $a$  lies in  $\mathfrak{T}_\mu^e$ , then  $\phi(a)$  is repelled from  $i_\alpha^\alpha(0) \in Z_{G_{\mathrm{aff}}}^\alpha$  under the following action of  $\mathbb{C}^*$  on  $Z_{G_{\mathrm{aff}}}^\alpha$ . Recall that  $Z_{G_{\mathrm{aff}}}^\alpha$  is a certain closure of the moduli space of  $G$ -bundles on  $\mathbb{P}^2$  (with homogeneous coordinates  $[z_0 : z_1 : z_2]$ ) trivialized at  $\ell_\infty$  (given by  $z_0 = 0$ ) and equipped with a reduction to  $B$  along  $\ell_0$  (given by  $z_2 = 0$ ). The Cartan torus  $T \subset B \subset G$  acts on  $Z_{G_{\mathrm{aff}}}^\alpha$  via trivialization at  $\ell_\infty$ , while  $\mathbb{C}_{\mathrm{vert}}^*$  acts on  $\mathbb{P}^2$  by  $c[z_0 : z_1 : z_2] = [z_0 : z_1 : cz_2]$ , and hence on  $Z_{G_{\mathrm{aff}}}^\alpha$  by transport of structure. Note that the action of  $\mathbb{C}_{\mathrm{vert}}^*$  lifts to the action on  $\widehat{\mathbb{P}}_k^2$  with the following property: if  $f$  is a non  $\mathbb{C}_{\mathrm{vert}}^*$ -fixed point on the exceptional divisor, then as  $c \in \mathbb{C}_{\mathrm{vert}}^*$  tends to infinity,  $c \cdot f$  tends to the singular point of the exceptional divisor. We consider the one-parametric subgroup  $\mathbb{C}^* \rightarrow T \times \mathbb{C}_{\mathrm{vert}}^* : c \mapsto (\bar{\rho}(c), c^h)$  where  $\bar{\rho}$  is the halfsum of positive coroots of  $G$  viewed as a cocharacter of  $T$ , while  $h$  is the Coxeter number of  $G$ . The desired action of  $\mathbb{C}^*$  on  $Z_{G_{\mathrm{aff}}}^\alpha$  is the action of this one-parametric subgroup. It

is clear that the only points of  $Z_{G_{\text{aff}}}^\alpha$  repelled from anything at all are the  $T$ -fixed points  $\mathbb{A}^\alpha \subset Z_{G_{\text{aff}}}^\alpha$ . It follows  $\phi(a) = i_\alpha^\alpha(0)$ .  $\square$

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